CS 5522: Artificial Intelligence II

Hidden Markov Models

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[These slides were adapted from CS188 Intro to AI at UC Berkeley. All materials available at http://ai.berkeley.edu.]
Pacman - Sonar (P4)
Video of Demo Pacman - Sonar (no beliefs)
Video of Demo Pacman - Sonar (no beliefs)
Video of Demo Pacman - Sonar (no beliefs)
Probability Recap
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- Conditional probability
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  \[ P(x|y) = \frac{P(x, y)}{P(y)} \]
- Product rule
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  $$P(x|y) = \frac{P(x, y)}{P(y)}$$

- Product rule

- Chain rule
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- Chain rule
Probability Recap

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- **Chain rule**
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- **Chain rule**
  \[ P(X_1, X_2, \ldots, X_n) = P(X_1)P(X_2|X_1)P(X_3|X_1, X_2)\ldots \]
  \[ = \prod_{i=1}^{n} P(X_i|X_1, \ldots, X_{i-1}) \]

- **X, Y independent if and only if:**
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- X, Y independent if and only if:

- X and Y are conditionally independent given Z if and only if:
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- **X, Y independent if and only if:** \( \forall x, y : P(x, y) = P(x)P(y) \)

- **X and Y are conditionally independent given Z if and only if:**
  \[ \forall x, y, z : P(x, y|z) = P(x|z)P(y|z) \]
Probability Recap

- Conditional probability
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  \[ P(X_1, X_2, \ldots, X_n) = P(X_1)P(X_2|X_1)P(X_3|X_1, X_2)\ldots = \prod_{i=1}^{n} P(X_i|X_1, \ldots, X_{i-1}) \]

- X, Y independent if and only if: \( \forall x, y : P(x, y) = P(x)P(y) \)

- X and Y are conditionally independent given Z if and only if: \( X \perp Y|Z \)
  \[ \forall x, y, z : P(x, y|z) = P(x|z)P(y|z) \]
Hidden Markov Models
Hidden Markov Models

- Markov chains not so useful for most agents
  - Need observations to update your beliefs

- Hidden Markov models (HMMs)
  - Underlying Markov chain over states $X$
  - You observe outputs (effects) at each time step

$X_1 \rightarrow X_2 \rightarrow X_3 \rightarrow X_4 \rightarrow \ldots$

$E_1 \rightarrow E_2 \rightarrow E_3 \rightarrow E_4 \rightarrow \ldots$
Example: Weather HMM

- An HMM is defined by:
  - Initial distribution: \( P(X_1) \)
  - Transitions: \( P(X_t \mid X_{t-1}) \)
  - Emissions: \( P(E_t \mid X_t) \)
Example: Weather HMM

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<table>
<thead>
<tr>
<th>( R_t )</th>
<th>( R_{t+1} )</th>
<th>( P(R_{t+1} \mid R_t) )</th>
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<tbody>
<tr>
<td>+r</td>
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Example: Weather HMM

An HMM is defined by:
- Initial distribution: $P(X_1)$
- Transitions: $P(X_t | X_{t-1})$  
- Emissions: $P(E_t | X_t)$

| $R_t$ | $R_{t+1}$ | $P(R_{t+1} | R_t)$ |
|-------|-----------|---------------------|
| +r    | +r        | 0.7                 |
| +r    | -r        | 0.3                 |
| -r    | +r        | 0.3                 |
| -r    | -r        | 0.7                 |

| $R_t$ | $U_t$ | $P(U_t | R_t)$ |
|-------|-------|----------------|
| +r    | +u    | 0.9            |
| +r    | -u    | 0.1            |
| -r    | +u    | 0.2            |
| -r    | -u    | 0.8            |
Example: Ghostbusters HMM

- $P(X_1) = \text{uniform}$

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$P(X_1)$
Example: Ghostbusters HMM

- $P(X_1) = \text{uniform}$
- $P(X|X') = \text{usually move clockwise, but sometimes move in a random direction or stay in place}$
Example: Ghostbusters HMM

- \( P(X_1) = \text{uniform} \)

- \( P(X|X') = \text{usually move clockwise, but sometimes move in a random direction or stay in place} \)
Example: Ghostbusters HMM

- \( P(X_1) = \text{uniform} \)

- \( P(X|X') = \) usually move clockwise, but sometimes move in a random direction or stay in place

- \( P(R_{ij} | X) = \) same sensor model as before: red means close, green means far away.

[Diagram showing the transition between states and sensor readings for Ghostbusters HMM]

\[
\begin{array}{c|c|c|c}
X_1 & X_2 & X_3 & X_4 \\
\hline
R_{i,j} & R_{i,j} & R_{i,j} & R_{i,j} \\
\end{array}
\]

\[
\begin{array}{c|c|c|c}
P(X_1) & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} \\
\frac{1}{9} & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} \\
\frac{1}{9} & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} \\
\end{array}
\]

\[
\begin{array}{c|c|c|c}
P(X|X'=\langle1,2\rangle) & \frac{1}{6} & \frac{1}{6} & \frac{1}{2} \\
0 & 1/6 & 0 \\
0 & 0 & 0 \\
\end{array}
\]
Video of Demo Ghostbusters - Circular Dynamics -- HMM
Video of Demo Ghostbusters - Circular Dynamics -- HMM
Video of Demo Ghostbusters - Circular Dynamics -- HMM
Joint Distribution of an HMM

- Joint distribution:
  \[ P(X_1, E_1, X_2, E_2, X_3, E_3) = P(X_1)P(E_1|X_1)P(X_2|X_1)P(E_2|X_2)P(X_3|X_2)P(E_3|X_3) \]

- More generally:
  \[ P(X_1, E_1, \ldots, X_T, E_T) = P(X_1)P(E_1|X_1) \prod_{t=2}^{T} P(X_t|X_{t-1})P(E_t|X_t) \]

- Questions to be resolved:
  - Does this indeed define a joint distribution?
  - Can every joint distribution be factored this way, or are we making some assumptions about the joint distribution by using this factorization?
From the chain rule, every joint distribution over $X_1, E_1, X_2, E_2, X_3, E_3$ can be written as:

$$P(X_1, E_1, X_2, E_2, X_3, E_3) = P(X_1)P(E_1|X_1)P(X_2|X_1, E_1)P(E_2|X_1, E_1, X_2)P(X_3|X_1, E_1, X_2, E_2)P(E_3|X_1, E_1, X_2, E_2, X_3)$$

Assuming that

$X_2 \perp E_1 \mid X_1, \quad E_2 \perp X_1, E_1 \mid X_2, \quad X_3 \perp X_1, E_1, E_2 \mid X_2, \quad E_3 \perp X_1, E_1, X_2, E_2 \mid X_3$

gives us the expression posited on the previous slide:

$$P(X_1, E_1, X_2, E_2, X_3, E_3) = P(X_1)P(E_1|X_1)P(X_2|X_1)P(E_2|X_2)P(X_3|X_2)P(E_3|X_3)$$
Chain Rule and HMMs

- From the chain rule, every joint distribution over $X_1, E_1, \ldots, X_T, E_T$ can be written as:
  \[ P(X_1, E_1, \ldots, X_T, E_T) = P(X_1)P(E_1|X_1) \prod_{t=2}^{T} P(X_t|X_1, E_1, \ldots, X_{t-1}, E_{t-1})P(E_t|X_1, E_1, \ldots, X_{t-1}, E_{t-1}, X_t) \]

- Assuming that for all $t$:
  - State independent of all past states and all past evidence given the previous state, i.e.:
    \[ X_t \perp X_1, E_1, \ldots, X_{t-2}, E_{t-2}, E_{t-1} \mid X_{t-1} \]
  - Evidence is independent of all past states and all past evidence given the current state, i.e.:
    \[ E_t \perp X_1, E_1, \ldots, X_{t-2}, E_{t-2}, X_{t-1}, E_{t-1} \mid X_t \]

gives us the expression posited on the earlier slide:
\[ P(X_1, E_1, \ldots, X_T, E_T) = P(X_1)P(E_1|X_1) \prod_{t=2}^{T} P(X_t|X_{t-1})P(E_t|X_t) \]
Many implied conditional independencies, e.g.,

\[ E_1 \perp X_2, E_2, X_3, E_3 \mid X_1 \]

To prove them

- Approach 1: follow similar (algebraic) approach to what we did in the Markov models lecture
- Approach 2: directly from the graph structure (3 lectures from now)
  - Intuition: If path between U and V goes through W, then \( U \perp V \mid W \)

[Some fineprint later]
Real HMM Examples

- **Speech recognition HMMs:**
  - Observations are acoustic signals (continuous valued)
  - States are specific positions in specific words (so, tens of thousands)

- **Machine translation HMMs:**
  - Observations are words (tens of thousands)
  - States are translation options

- **Robot tracking:**
  - Observations are range readings (continuous)
  - States are positions on a map (continuous)
Filtering / Monitoring

- Filtering, or monitoring, is the task of tracking the distribution $B_t(X) = P_t(X_t | e_1, ..., e_t)$ (the belief state) over time.

- We start with $B_1(X)$ in an initial setting, usually uniform.

- As time passes, or we get observations, we update $B(X)$.

- The Kalman filter was invented in the 60’s and first implemented as a method of trajectory estimation for the Apollo program.
Example: Robot Localization

Sensor model: can read in which directions there is a wall, never more than 1 mistake

Motion model: may not execute action with small prob.
Example: Robot Localization

Lighter grey: was possible to get the reading, but less likely b/c required 1 mistake
Example: Robot Localization

Lighter grey: was possible to get the reading, but less likely
b/c required 1 mistake
Example: Robot Localization
Example: Robot Localization

\[ t=2 \]
Example: Robot Localization

Prob

0 1

t=3
Example: Robot Localization

$t=3$

Prob

0

1
Example: Robot Localization

\[ t=4 \]

\[ \text{Prob} \]

0

1
Example: Robot Localization

$\text{Prob}$

$0$ $1$

$t=4$
Example: Robot Localization

Prob

0 1

t=5
Inference: Base Cases

\[ P(X_1|e_1) \]
Inference: Base Cases

\[ P(X_1|e_1) \]

\[
P(x_1|e_1) = P(x_1, e_1)/P(e_1)
\]

\[
\propto_{x_1} P(x_1, e_1)
\]

\[
= P(x_1)P(e_1|x_1)
\]
Inference: Base Cases

\[ P(X_1|e_1) \]

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\[ \propto_{x_1} P(x_1, e_1) \]
\[ = P(x_1)P(e_1|x_1) \]

\[ P(X_2) \]

\[ P(x_2) = \sum_{x_1} P(x_1, x_2) \]
\[ = \sum_{x_1} P(x_1)P(x_2|x_1) \]
Passage of Time

- Assume we have current belief $P(X \mid \text{evidence to date})$

$$B(X_t) = P(X_t|e_{1:t})$$
Passage of Time

- Assume we have current belief $P(X \mid \text{evidence to date})$
  
  $$B(X_t) = P(X_t|e_{1:t})$$

- Then, after one time step passes:
Passage of Time

- Assume we have current belief $P(X \mid \text{evidence to date})$
  
  $$B(X_t) = P(X_t \mid e_{1:t})$$

- Then, after one time step passes:
  
  $$P(X_{t+1} \mid e_{1:t})$$
Passage of Time

- Assume we have current belief $P(X \mid \text{evidence to date})$
  \[
  B(X_t) = P(X_t \mid e_{1:t})
  \]
- Then, after one time step passes:
  \[
  P(X_{t+1} \mid e_{1:t}) = \sum_{x_t} P(X_{t+1}, x_t \mid e_{1:t})
  \]
Passage of Time

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  \]

  \[= \sum_{x_t} P(X_{t+1} \mid x_t, e_{1:t}) P(x_t \mid e_{1:t}) \]
Passage of Time

- Assume we have current belief $P(X \mid \text{evidence to date})$
  
  $B(X_t) = P(X_t|e_{1:t})$

- Then, after one time step passes:

  $$P(X_{t+1}|e_{1:t}) = \sum_{x_t} P(X_{t+1}, x_t|e_{1:t})$$

  $$= \sum_{x_t} P(X_{t+1}|x_t, e_{1:t})P(x_t|e_{1:t})$$

  $$= \sum_{x_t} P(X_{t+1}|x_t)P(x_t|e_{1:t})$$
Passage of Time

- Assume we have current belief $P(X \mid \text{evidence to date})$

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- Then, after one time step passes:

$$P(X_{t+1}|e_{1:t}) = \sum_{x_t} P(X_{t+1}, x_t|e_{1:t})$$

$$= \sum_{x_t} P(X_{t+1}|x_t, e_{1:t})P(x_t|e_{1:t})$$

$$= \sum_{x_t} P(X_{t+1}|x_t)P(x_t|e_{1:t})$$

- Or compactly:

$$B'(X_{t+1}) = \sum_{x_t} P(X'|x_t)B(x_t)$$
Passage of Time

- Assume we have current belief \( P(X \mid \text{evidence to date}) \)

\[
B(X_t) = P(X_t \mid e_{1:t})
\]

- Then, after one time step passes:

\[
P(X_{t+1} \mid e_{1:t}) = \sum_{x_t} P(X_{t+1}, x_t \mid e_{1:t})
\]

\[
= \sum_{x_t} P(X_{t+1} \mid x_t, e_{1:t}) P(x_t \mid e_{1:t})
\]

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= \sum_{x_t} P(X_{t+1} \mid x_t) P(x_t \mid e_{1:t})
\]

- Or compactly:

\[
B'(X_{t+1}) = \sum_{x_t} P(X' \mid x_t) B(x_t)
\]

- Basic idea: beliefs get “pushed” through the transitions
  
  - With the “B” notation, we have to be careful about what time step \( t \) the belief is about, and what evidence it includes
As time passes, uncertainty “accumulates”

(Transition model: ghosts usually go clockwise)

\[ T = 1 \]
As time passes, uncertainty “accumulates”

(Transition model: ghosts usually go clockwise)
Example: Passage of Time

- As time passes, uncertainty “accumulates”

(Transition model: ghosts usually go clockwise)
As time passes, uncertainty "accumulates"

(Transition model: ghosts usually go clockwise)
Assume we have current belief $P(X \mid \text{previous evidence})$:

$$B'(X_{t+1}) = P(X_{t+1} \mid e_{1:t})$$
Observation

- Assume we have current belief \( P(X \mid \text{previous evidence}) \):
  \[
  B'(X_{t+1}) = P(X_{t+1} \mid e_{1:t})
  \]
- Then, after evidence comes in:
Observation

- Assume we have current belief $P(X \mid \text{previous evidence})$:
  \[ B'(X_{t+1}) = P(X_{t+1} \mid e_{1:t}) \]
- Then, after evidence comes in:
  \[ P(X_{t+1} \mid e_{1:t+1}) = \]
Assume we have current belief $P(X \mid \text{previous evidence})$:

$$B'(X_{t+1}) = P(X_{t+1}|e_{1:t})$$

Then, after evidence comes in:

$$P(X_{t+1}|e_{1:t+1}) = \frac{P(X_{t+1}, e_{t+1}|e_{1:t})}{P(e_{t+1}|e_{1:t})}$$
Assume we have current belief $P(X \mid \text{previous evidence})$:

$$B'(X_{t+1}) = P(X_{t+1} \mid e_{1:t})$$

Then, after evidence comes in:

$$P(X_{t+1} \mid e_{1:t+1}) = \frac{P(X_{t+1}, e_{t+1} \mid e_{1:t})}{P(e_{t+1} \mid e_{1:t})}$$

$$\propto_{X_{t+1}} P(X_{t+1}, e_{t+1} \mid e_{1:t})$$
**Observation**

- Assume we have current belief $P(X \mid \text{previous evidence})$:
  
  $$ B'(X_{t+1}) = P(X_{t+1} \mid e_{1:t}) $$

- Then, after evidence comes in:
  
  $$ P(X_{t+1} \mid e_{1:t+1}) = \frac{P(X_{t+1}, e_{t+1} \mid e_{1:t})}{P(e_{t+1} \mid e_{1:t})} $$
  
  $$ \propto P(X_{t+1}, e_{t+1} \mid e_{1:t}) $$
  
  $$ = P(e_{t+1} \mid e_{1:t}, X_{t+1}) P(X_{t+1} \mid e_{1:t}) $$
Assume we have current belief $P(X \mid \text{previous evidence})$:

$$B'(X_{t+1}) = P(X_{t+1} \mid e_{1:t})$$

Then, after evidence comes in:

$$P(X_{t+1} \mid e_{1:t+1}) = \frac{P(X_{t+1}, e_{t+1} \mid e_{1:t})}{P(e_{t+1} \mid e_{1:t})}$$

$$\propto_{X_{t+1}} P(X_{t+1}, e_{t+1} \mid e_{1:t})$$

$$= P(e_{t+1} \mid e_{1:t}, X_{t+1}) P(X_{t+1} \mid e_{1:t})$$

$$= P(e_{t+1} \mid X_{t+1}) P(X_{t+1} \mid e_{1:t})$$
Assume we have current belief $P(X \mid \text{previous evidence})$:

$$B'(X_{t+1}) = P(X_{t+1} \mid e_{1:t})$$

Then, after evidence comes in:

$$P(X_{t+1} \mid e_{1:t+1}) = \frac{P(X_{t+1}, e_{t+1} \mid e_{1:t})}{P(e_{t+1} \mid e_{1:t})} \propto_{X_{t+1}} P(X_{t+1}, e_{t+1} \mid e_{1:t})$$

$$= P(e_{t+1} \mid e_{1:t}, X_{t+1})P(X_{t+1} \mid e_{1:t})$$

$$= P(e_{t+1} \mid X_{t+1})P(X_{t+1} \mid e_{1:t})$$

Or, compactly:
Observation

- Assume we have current belief \( P(X | \text{previous evidence}) \):
  \[
  B'(X_{t+1}) = P(X_{t+1} | e_{1:t})
  \]

- Then, after evidence comes in:
  \[
  P(X_{t+1} | e_{1:t+1}) = \frac{P(X_{t+1}, e_{t+1} | e_{1:t})}{P(e_{t+1} | e_{1:t})}
  \]
  \[
  \propto P(X_{t+1}, e_{t+1} | e_{1:t})
  \]
  \[
  = P(e_{t+1} | e_{1:t}, X_{t+1}) P(X_{t+1} | e_{1:t})
  \]
  \[
  = P(e_{t+1} | X_{t+1}) P(X_{t+1} | e_{1:t})
  \]
- Or, compactly:
  \[
  B(X_{t+1}) \propto P(e_{t+1} | X_{t+1}) B'(X_{t+1})
  \]
Assume we have current belief $P(X \mid \text{previous evidence})$:

$$B'(X_{t+1}) = P(X_{t+1} \mid e_{1:t})$$

Then, after evidence comes in:

$$P(X_{t+1} \mid e_{1:t+1}) = \frac{P(X_{t+1}, e_{t+1} \mid e_{1:t})}{P(e_{t+1} \mid e_{1:t})} \propto P(X_{t+1}, e_{t+1} \mid e_{1:t})$$

$$= P(e_{t+1} \mid e_{1:t}, X_{t+1}) P(X_{t+1} \mid e_{1:t})$$

$$= P(e_{t+1} \mid X_{t+1}) P(X_{t+1} \mid e_{1:t})$$

Or, compactly:

$$B(X_{t+1}) \propto P(e_{t+1} \mid X_{t+1}) B'(X_{t+1})$$

Basic idea: beliefs “reweighted” by likelihood of evidence

Unlike passage of time, we have to renormalize
Example: Observation

- As we get observations, beliefs get reweighted, uncertainty “decreases”

Before observation

After observation

\[ B(X) \propto P(e|X)B'(X) \]
Example: Weather HMM

- \( B(+r) = 0.5 \)
  - \( B'(+r) = 0.5 \)
  - \( B(+r) = 0.818 \)
  - \( B'(+r) = 0.627 \)
  - \( B(+r) = 0.883 \)
- \( B(-r) = 0.5 \)
  - \( B'(-r) = 0.5 \)
  - \( B(-r) = 0.182 \)
  - \( B'(-r) = 0.373 \)
  - \( B(-r) = 0.117 \)
Example: Weather HMM

\[ P(R_{t+1} | R_t) \]

\[ \begin{array}{c|ccc}
 R_t & R_{t+1} & P(R_{t+1} | R_t) \\
 \hline
 +r & +r & 0.7 \\
 +r & -r & 0.3 \\
 -r & +r & 0.3 \\
 -r & -r & 0.7 \\
\end{array} \]
Example: Weather HMM

\[ R_t \rightarrow R_{t+1} \rightarrow R_{t+2} \]

- \( B(+r) = 0.5 \)
- \( B(-r) = 0.5 \)

- \( B(+r) = 0.818 \)
- \( B(-r) = 0.182 \)

- \( B(+r) = 0.883 \)
- \( B(-r) = 0.117 \)

<table>
<thead>
<tr>
<th>( R_t )</th>
<th>( R_{t+1} )</th>
<th>( P(R_{t+1} \mid R_t) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>+r</td>
<td>+r</td>
<td>0.7</td>
</tr>
<tr>
<td>+r</td>
<td>-r</td>
<td>0.3</td>
</tr>
<tr>
<td>-r</td>
<td>+r</td>
<td>0.3</td>
</tr>
<tr>
<td>-r</td>
<td>-r</td>
<td>0.7</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>( R_t )</th>
<th>( U_t )</th>
<th>( P(U_t \mid R_t) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>+r</td>
<td>+u</td>
<td>0.9</td>
</tr>
<tr>
<td>+r</td>
<td>-u</td>
<td>0.1</td>
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<tr>
<td>-r</td>
<td>+u</td>
<td>0.2</td>
</tr>
<tr>
<td>-r</td>
<td>-u</td>
<td>0.8</td>
</tr>
</tbody>
</table>
The Forward Algorithm

- We are given evidence at each time and want to know
  \[ B_t(X) = P(X_t|e_{1:t}) \]

- We can derive the following updates
  \[ P(x_t|e_{1:t}) \propto_X P(x_t, e_{1:t}) \]

  We can normalize as we go if we want to have \( P(x|e) \) at each time step, or just once at the end...
The Forward Algorithm

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\[ B_t(X) = P(X_t|e_{1:t}) \]

- We can derive the following updates

\[
P(x_t|e_{1:t}) \propto \sum_{x_{t-1}} P(x_{t-1}, x_t, e_{1:t})
\]


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= \sum_{x_{t-1}} P(x_{t-1}, e_{1:t-1}) P(x_t|x_{t-1}) P(e_t|x_t)
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\]

We can normalize as we go if we want to have \( P(x|e) \) at each time step, or just once at the end...
Online Belief Updates

- Every time step, we start with current $P(X \mid \text{evidence})$
- We update for time:
  $$P(x_t|e_{1:t-1}) = \sum_{x_{t-1}} P(x_{t-1}|e_{1:t-1}) \cdot P(x_t|x_{t-1})$$
- We update for evidence:
  $$P(x_t|e_{1:t}) \propto_{X} P(x_t|e_{1:t-1}) \cdot P(e_t|x_t)$$
- The forward algorithm does both at once (and doesn’t normalize)
Pacman - Sonar (P4)
Video of Demo Pacman - Sonar (with beliefs)
Video of Demo Pacman - Sonar (with beliefs)
Video of Demo Pacman - Sonar (with beliefs)
Next Time: Particle Filtering and Applications of HMMs