Logistic Regression

Some slides adapted from Dan Jurfasky and Brendan O'Connor

Naïve Bayes Recap

- Bag of words (order independent)

- Features are assumed independent given class

$$P(x_1, \ldots, x_n | c) = P(x_1 | c) \ldots P(x_n | c)$$

Q: Is this really true?

The problem with assuming conditional independence

- Correlated features -> double counting evidence
  - Parameters are estimated independently

- This can hurt classifier accuracy and calibration

Logistic Regression

- (Log) Linear Model – similar to Naïve Bayes
- Doesn’t assume features are independent
- Correlated features don’t “double count”

What are “Features”?

- A feature function, f
  - Input: Document, D (a string)
  - Output: Feature Vector, X

What are “Features”?

$$f(d) = \begin{pmatrix} \text{count(“boring”)} \\ \text{count(“not boring”)} \\ \text{length of document} \\ \text{author of document} \\ \vdots \end{pmatrix}$$

Doesn’t have to be just “bag of words”

Feature Templates

- Typically “feature templates” are used to generate many features at once

- For each word:
  - ${w}_count
  - ${w}_lowercase
  - ${w}_with_NOT_before_count

Logistic Regression: Example

- Compute Features:

$$f(d_i) = x_i = \begin{pmatrix} \text{count(``nigerian'')} \\ \text{count(``prince'')} \\ \text{count(``nigerian prince'')} \end{pmatrix}$$

- Assume we are given some weights:

$$w = \begin{pmatrix} -1.0 \\ -1.0 \\ 4.0 \end{pmatrix}$$

Logistic Regression: Example

- Compute Features
- We are given some weights
- Compute the dot product:

$$z = \sum_{i=0}^{|X|} w_i x_i$$

Logistic Regression: Example

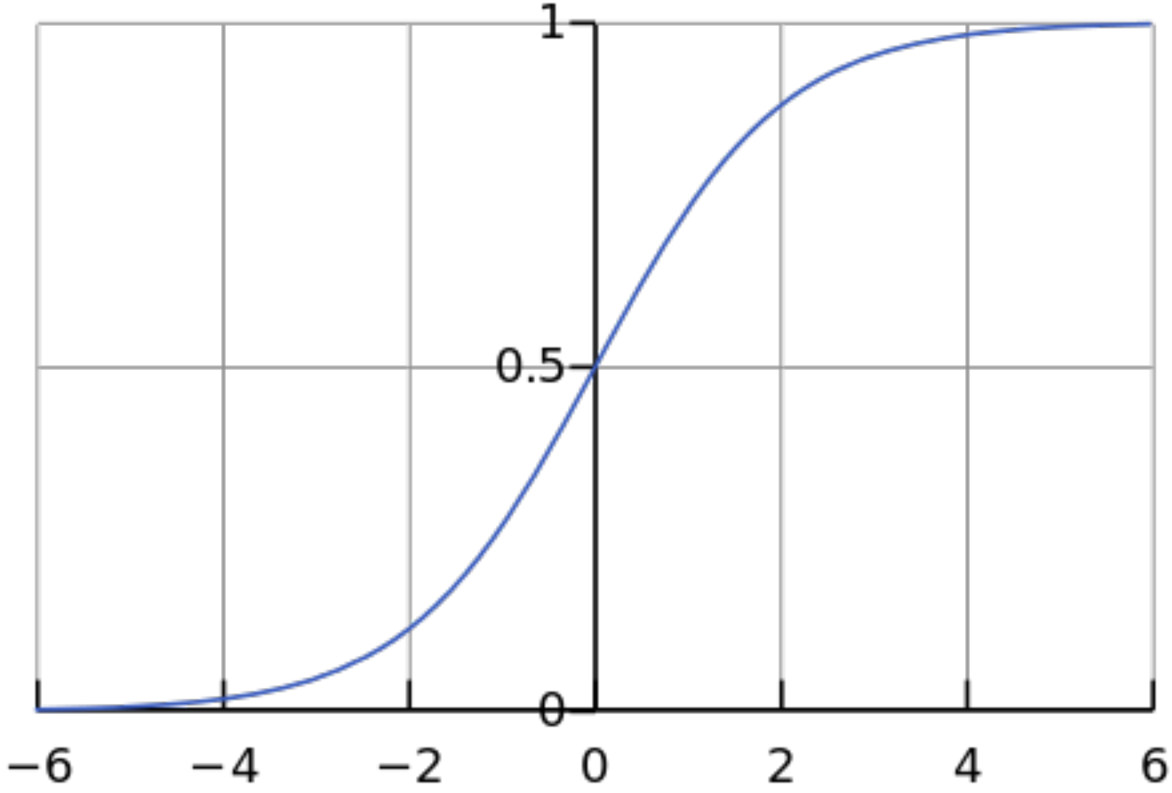
- Compute the dot product:

$$z = \sum_{i=0}^{|X|} w_i x_i$$

- Compute the logistic function:

$$P(\text{spam}|x) = \frac{e^z}{e^z + 1} = \frac{1}{1 + e^{-z}}$$

The Logistic function

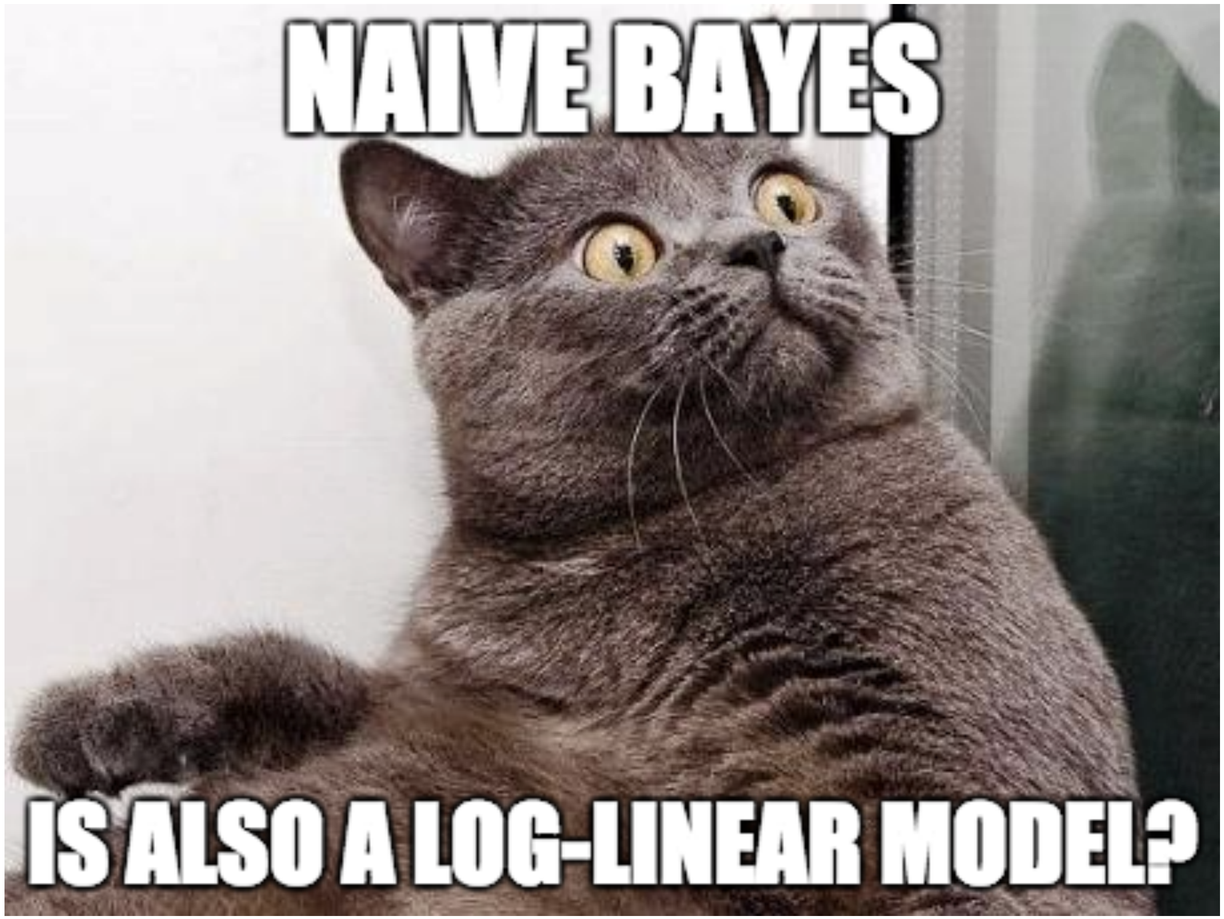


$$P(\text{spam}|x) = \frac{e^z}{e^z + 1} = \frac{1}{1 + e^{-z}}$$

The Dot Product

$$z = \sum_{i=0}^{|X|} w_i x_i$$

- Intuition: weighted sum of features
- All Linear models have this form


NAIVE BAYES
IS ALSO A LOG-LINEAR MODEL?

Naïve Bayes as a log-linear model

- Q: what are the features?

- Q: what are the weights?

Naïve Bayes as a Log-Linear Model

$$P(\text{spam}|D) \propto P(\text{spam}) \prod_{w \in D} P(w|\text{spam})$$

$$P(\text{spam}|D) \propto P(\text{spam}) \prod_{w \in \text{Vocab}} P(w|\text{spam})^{x_i}$$

$$\log P(\text{spam}|D) \propto \log P(\text{spam}) + \sum_{w \in \text{Vocab}} x_i \cdot \log P(w|\text{spam})$$

Naïve Bayes as a Log-Linear Model

$$\log P(\text{spam}|D) \propto \log P(\text{spam}) + \sum_{w \in \text{Vocab}} x_i \cdot \log P(w|\text{spam})$$

In both Naïve Bayes and Logistic Regression we Compute The Dot Product!

NB vs. LR

- Both compute the dot product
- NB: sum of log probabilities
- LR: logistic function

NB vs. LR: Parameter Learning

- Naïve Bayes:
  - Learn conditional probabilities **independently** by counting

- Logistic Regression:
  - Learn weights **jointly**

LR: Learning Weights

- Given: a set of feature vectors and labels
- Goal: learn the weights

Learning Weights

$$
\begin{array}{cc}
 & \begin{array}{ccccc} \phantom{x_{11}} & \phantom{x_{12}} & \text{Feature} & & \phantom{x_{1n}} \end{array} \\
\text{Document} \rightarrow & \begin{bmatrix}
x_{11} & x_{12} & x_{13} & \cdots & x_{1n} \\
x_{21} & x_{22} & x_{23} & \cdots & x_{2n} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
x_{d1} & x_{d2} & x_{d3} & \cdots & x_{dn}
\end{bmatrix}
\end{array}
\quad
\begin{array}{c}
\text{Labels} \\
\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}
\end{array}
$$

Q: what parameters should we choose?

- What is the right value for the weights?

- Maximum Likelihood Principle:
  - Pick the parameters that maximize the probability of the data

Maximum Likelihood Estimation

$$w_{\mathrm{MLE}} = \operatorname{argmax}_w \log P(y_1, \ldots, y_d | x_1, \ldots, x_d; w)$$

$$= \operatorname{argmax}_w \sum_i \log P(y_i | x_i; w)$$

$$= \operatorname{argmax}_w \sum_i \log \begin{cases} p_i, & \text{if } y_i = 1 \\ 1 - p_i, & \text{if } y_i = 0 \end{cases}$$

$$= \operatorname{argmax}_w \sum_i \log p_i^{\mathbb{I}(y_i=1)} (1 - p_i)^{\mathbb{I}(y_i=0)}$$

Maximum Likelihood Estimation

$$= \operatorname{argmax}_w \sum_i \log p_i^{\mathbb{I}(y_i=1)}(1-p_i)^{\mathbb{I}(y_i=0)}$$

$$= \operatorname{argmax}_w \sum_i y_i \log p_i + (1-y_i)\log(1-p_i)$$

Maximum Likelihood Estimation

- Unfortunately there is no closed form solution
  - (like there was with naïve bayes)
- Solution:
  - Iteratively climb the log-likelihood surface through the derivatives for each weight
- Luckily, the derivatives turn out to be nice

Gradient ascent

Loop While not converged:

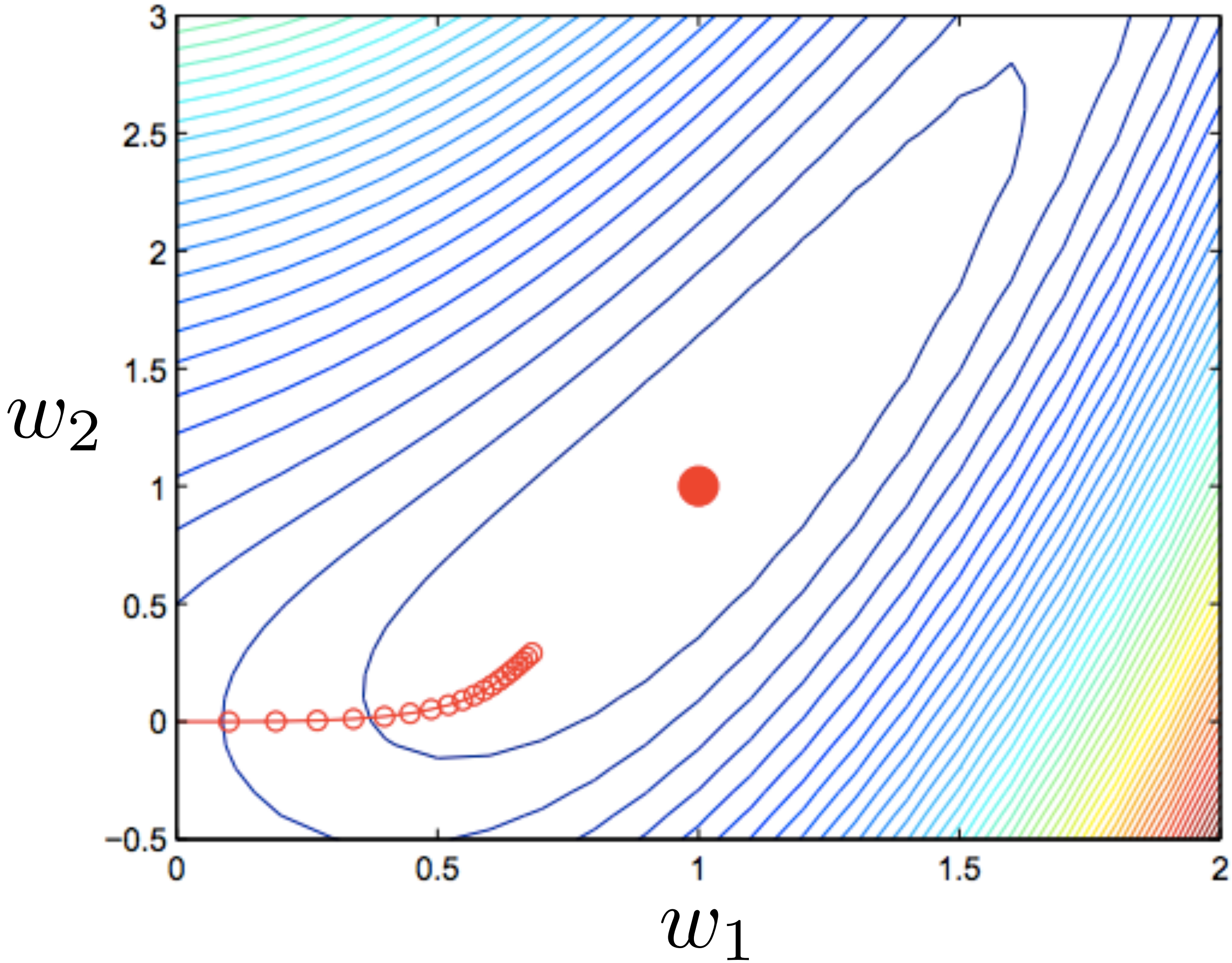
For all features **j**, compute and add derivatives

$$w_j^{\text{new}} = w_j^{\text{old}} + \eta \frac{\partial}{\partial w_j} \mathcal{L}(w)$$

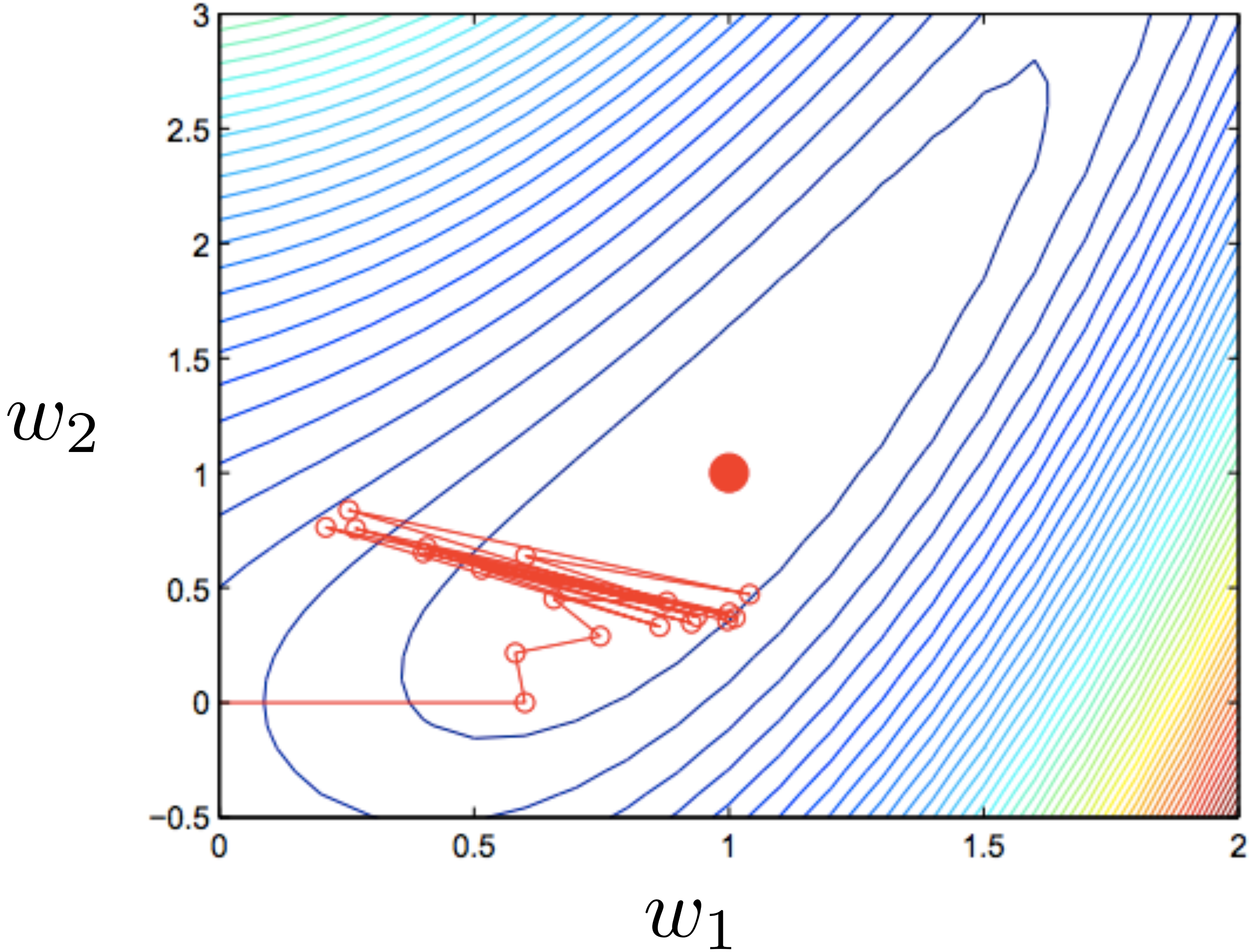
$\mathcal{L}(w)$: Training set log-likelihood

$\left(\frac{\partial \mathcal{L}}{\partial w_1}, \frac{\partial \mathcal{L}}{\partial w_2}, \ldots, \frac{\partial \mathcal{L}}{\partial w_n} \right)$: Gradient vector

Gradient ascent



Gradient ascent



LR Gradient

$$\frac{\partial \mathcal{L}}{\partial w_j} = \sum_i (y_i - p_i) x_j$$

Derivative of Sigmoid

$$\frac{ds(x)}{dx} = \frac{1}{1+e^{-x}}$$

$$= \left(\frac{1}{1+e^{-x}}\right)^2 \frac{d}{dx}(1+e^{-x})$$

$$= \left(\frac{1}{1+e^{-x}}\right)^2 e^{-x}(-1)$$

$$= \left(\frac{1}{1+e^{-x}}\right)\left(\frac{1}{1+e^{-x}}\right)(-e^{-x})$$

$$= \left(\frac{1}{1+e^{-x}}\right)\left(\frac{-e^{-x}}{1+e^{-x}}\right)$$

$$= s(x)(1-s(x))$$

Logistic Regression: Pros and Cons

- Doesn't assume conditional independence of features
  - Better calibrated probabilities
  - Can handle highly correlated overlapping features
- NB is faster to train, less likely to overfit

NB & LR

- Both are linear models

$$z = \sum_{i=0}^{|X|} w_i x_i$$

- Training is different:
  - NB: weights are trained independently
  - LR: weights trained jointly

Perceptron Algorithm

- Very sin
- Not exa

Perceptron Algorithm

- Algorithm is Very similar to logistic regression
- Not exactly computing gradients

```
Initalize weight vector w = 0
Loop for K iterations
    Loop For all training examples x_i
        if sign(w * x_i) != y_i
            w += (y_i - sign(w * x_i)) * x_i
```

Differences between LR and Perceptron

- Online learning vs. Batch

- Perceptron doesn’t always make updates

MultiClass Classification

- Q: what if we have more than 2 categories?
  - Sentiment: Positive, Negative, Neutral
  - Document topics: Sports, Politics, Business, Entertainment, …
- Could train a seperate logistic regression model for each category...
- Pretty clear what to do with Naive Bayes.

Log-Linear Models

$$P(y|x) \propto e^{w \cdot f(d,y)}$$

$$P(y|x) = \frac{1}{Z(w)} e^{w \cdot f(d,y)}$$

MultiClass Logistic Regression

$$P(y|x) \propto e^{w \cdot f(d,y)}$$

$$P(y|x) = \frac{1}{Z(w)} e^{w \cdot f(d,y)}$$

$$\boxed{P(y|x) = \frac{e^{w \cdot f(d,y)}}{\sum_{y' \in Y} e^{w \cdot f(d,y')}}}$$

MultiClass Logistic Regression

- Binary logistic regression:
  - We have one feature vector that matches the size
    o
- Mu
  - o

Can represent this in practice with one giant weight vector and repeated features for each category.

$$w_{\text{pos}} \qquad w_{\text{neg}} \qquad w_{\text{neut}}$$

Q: How to compute posterior class probabilities for multiclass?

$$P(y = j | x_i) = \frac{e^{w_j \cdot x_i}}{\sum_k e^{w_k \cdot x_i}}$$

Maximum Likelihood Estimation

$$w_{\mathrm{MLE}} = \mathrm{argmax}_w \log P(y_1, \ldots, y_n | x_1, \ldots, x_n; w)$$

$$= \mathrm{argmax}_w \sum_i \log P(y_i | x_i; w)$$

$$= \mathrm{argmax}_w \sum_i \log \frac{e^{w \cdot f(x_i, y_i)}}{\sum_{y' \in Y} e^{w \cdot f(x_i, y_i)}}$$

Multiclass LR Gradient

$$\frac{\partial \mathcal{L}}{\partial w_j} = \sum_{i=1}^{D} f_j(y_i, d_i) - \sum_{i=1}^{D} \sum_{y \in Y} f_j(y, d_i) P(y|d_i)$$

MAP-based learning (perceptron)

$$\frac{\partial \mathcal{L}}{\partial w_j} = \sum_{i=1}^{D} f_j(y_i, d_i) - \sum_{i=1}^{D} f_j(\arg\max_{y \in Y} P(y|d_i), d_i)$$

Online Learning (perceptron)

- Rather than making a full pass through the data, compute gradient and update parameters after each training example.

MultiClass Perceptron Algorithm

```
Initalize weight vector w = 0
Loop for K iterations
    Loop For all training examples x_i
        y_pred = argmax_y w_y * x_i
        if y_pred != y_i
            w_y_gold += x_i
            w_y_pred -= x_i
```

Q: what if there are only 2 categories?

$$P(y = j|x_i) = \frac{e^{w_j \cdot x_i}}{\sum_k e^{w_k \cdot x_i}}$$

Q: what if there are only 2 categories?

$$P(y=1|x) = \frac{e^{w_1 \cdot x}}{e^{w_0 \cdot x + w_1 \cdot x - w_1 \cdot x} + e^{w_1 \cdot x}}$$

Q: what if there are only 2 categories?

$$P(y = 1|x) = \frac{e^{w_1 \cdot x}}{e^{w_0 \cdot x - w_1 \cdot x} e^{w_1 \cdot x} + e^{w_1 \cdot x}}$$

Q: what if there are only 2 categories?

$$P(y=1|x) = \frac{e^{w_1 \cdot x}}{e^{w_1 \cdot x}(e^{w_0 \cdot x - w_1 \cdot x} + 1)}$$

Q: what if there are only 2 categories?

$$P(y = 1|x) = \frac{1}{e^{w_0 \cdot x - w_1 \cdot x} + 1}$$

Q: what if there are only 2 categories?

$$P(y = 1|x) = \frac{1}{e^{-w' \cdot x} + 1}$$

Regularization

- Combating over fitting

- Intuition: don't let the weights get very large

$$w_{\mathrm{MLE}} = \mathrm{argmax}_w \log P(y_1, \ldots, y_d | x_1, \ldots, x_d; w)$$

$$\mathrm{argmax}_w \log P(y_1, \ldots, y_d | x_1, \ldots, x_d; w) - \delta \sum_{i=1}^{V} w_i^2$$

Regularization in the Perceptron Algorithm

- Can’t directly include regularization in gradient

- # of iterations

- Parameter averaging