Hidden Markov Models

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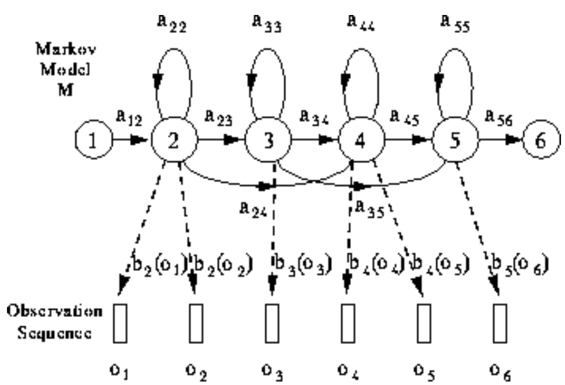


Sequences of R.V.s

- Previously we assumed IID data

$$P(x_1, x_2, x_3, \dots, x_n)$$
$$= P(x_1)P(x_2)P(x_3)\dots P(x_n)$$

- This is a useful assumption
  - Makes inference easy
- But, often too restrictive
  - E.g. Sequences of words not really independent
- Q: how can we introduce some dependence without blowing up inference and #parameters?

(**non-**hidden) Markov Models

- Answer: Markov Assumption

$$P(x_k|x_1, x_2, x_3, \ldots, x_{k-1}) = P(x_k|x_{k-1})$$

- Entire history is captured by previous state

$$P(x_1, x_2, x_3, \ldots, x_n)$$

$$= P(x_1)P(x_2|x_1)P(x_3|x_1, x_2)\ldots P(x_n|x_1, x_2, x_3, \ldots, x_{n-1})$$

$$= P(x_1)P(x_2|x_1)P(x_3|x_2)\ldots P(x_n|x_{n-1})$$

Application: Language Modeling

- Random variables:
  - Sequences of words or characters
- Estimate transition probabilities from data using maximum likelihood
- State space: all English words
- IID Assumption => unigram language model
- First-order Markov Model => bigram LM
- Second-order => trigram LM

Application: Language Modeling

- Unigram LM:

$$P(x_k|x_1, x_2, x_3, \ldots, x_{k-1}) = P(x_k)$$

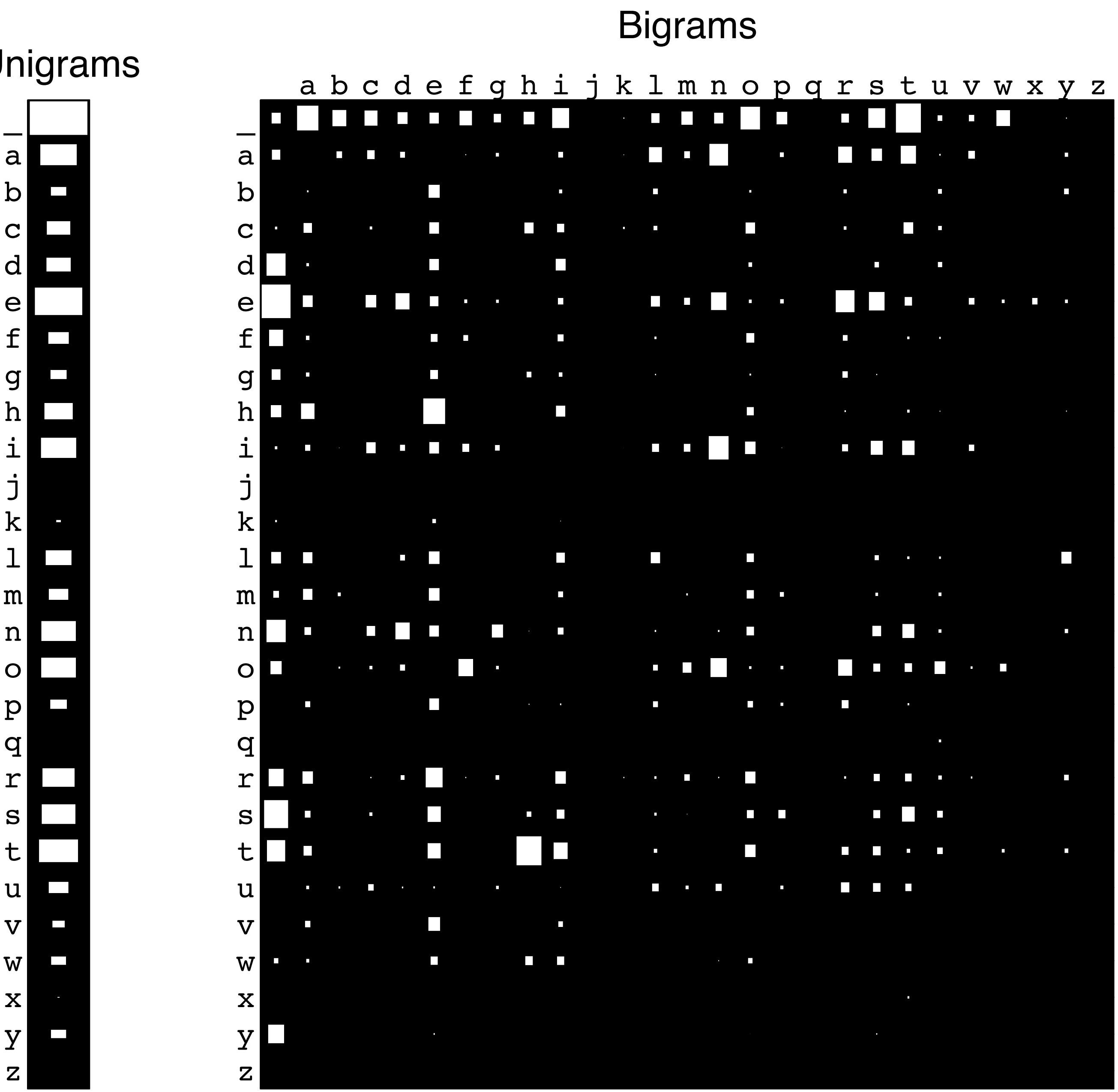
- Bigram LM (First-order Markov Model):

$$P(x_k|x_1, x_2, x_3, \ldots, x_{k-1}) = P(x_k|x_{k-1})$$

- Trigram LM (Second-order Markov Model):

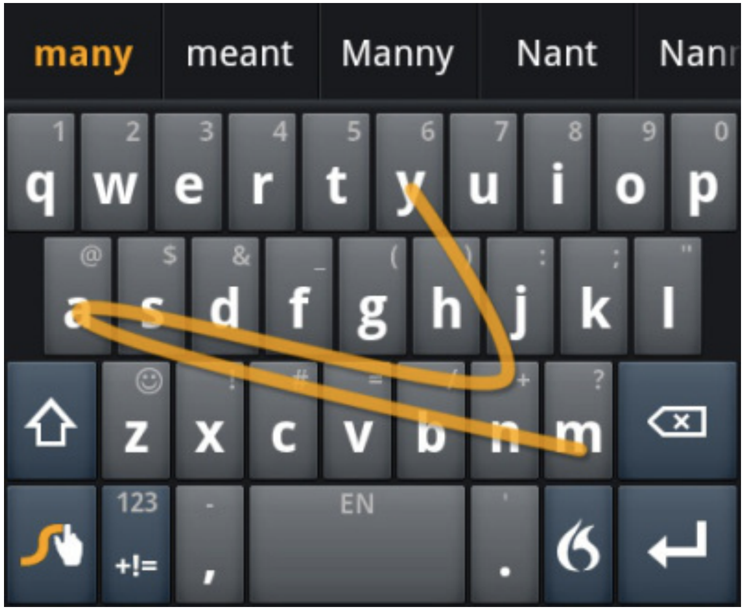
$$P(x_k|x_1, x_2, x_3, \ldots, x_{k-1}) = P(x_k|x_{k-1}, x_{k-2})$$

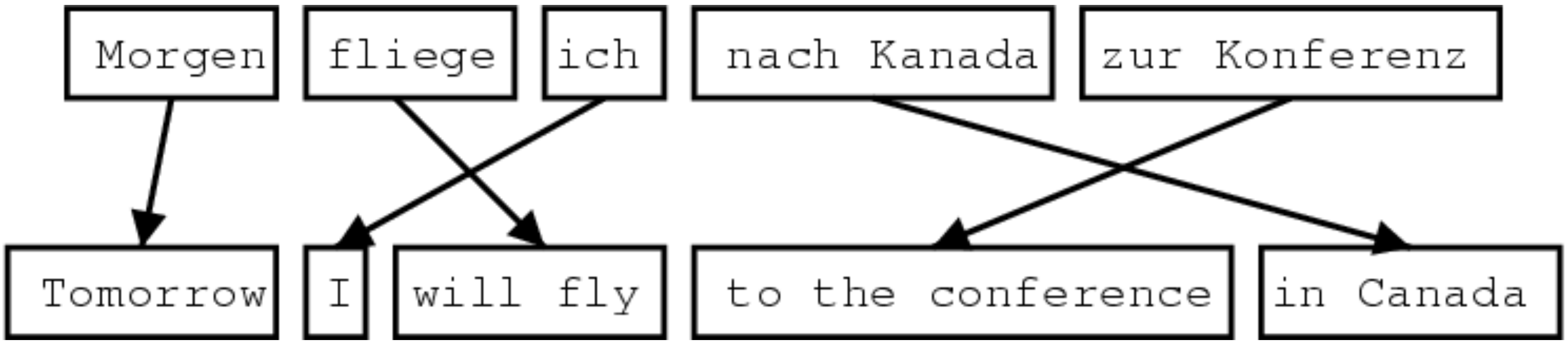
1	0.16098	_
2	0.06687	a
3	0.01414	b
4	0.02938	c
5	0.03107	d
6	0.11055	e
7	0.02325	f
8	0.01530	g
9	0.04174	h
10	0.06233	i
11	0.00060	j
12	0.00309	k
13	0.03515	l
14	0.02107	m
15	0.06007	n
16	0.06066	o
17	0.01594	p
18	0.00077	q
19	0.05265	r
20	0.05761	s
21	0.07566	t
22	0.02149	u
23	0.00993	v
24	0.01341	w
25	0.00208	x
26	0.01381	y
27	0.00039	z



What is Language Modeling Used For?

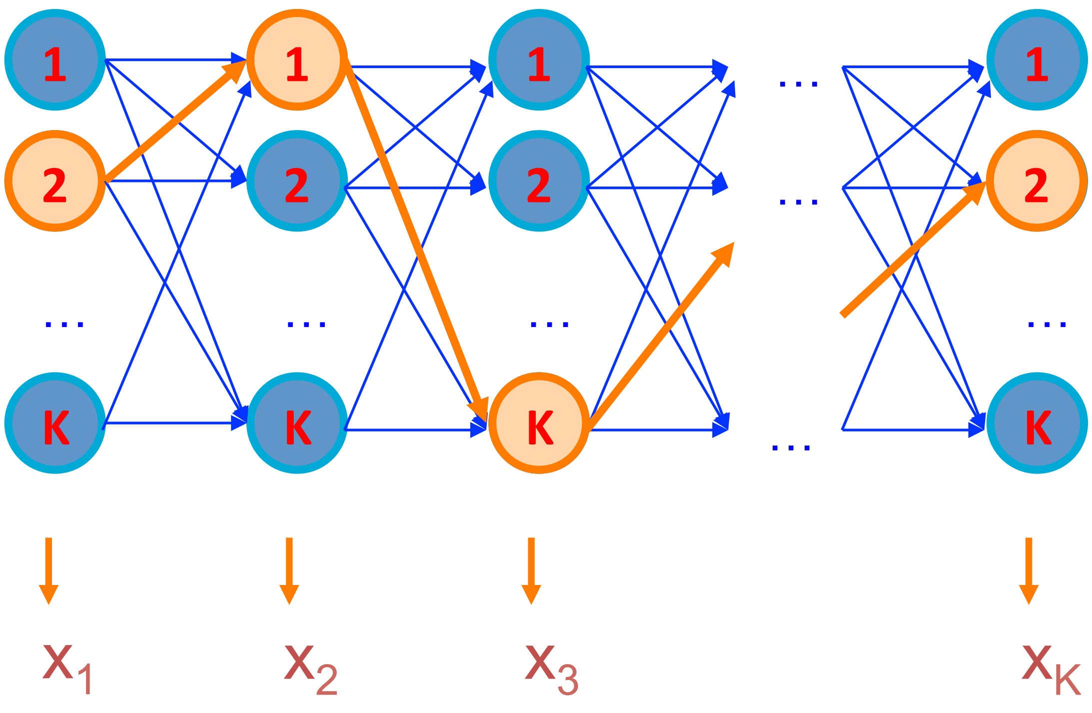
- Sentence Completion
  - Predictive Text Input
- Classification
  - Naïve bayes == unigram
- Machine Translation



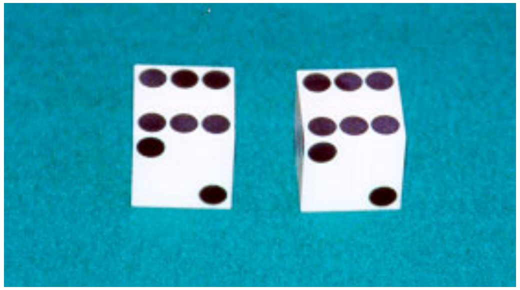


Hidden Markov Models

(Slides from Pedro Domingos)



Example: The dishonest casino



A casino has two dice:

- Fair die
  $P(1) = P(2) = P(3) = P(4) = P(5) = P(6) = 1/6$
- Loaded die
  $P(1) = P(2) = P(3) = P(4) = P(5) = 1/10$
  $P(6) = 1/2$

Casino player switches from
fair to loaded die with probability 1/20 at each turn

Game:

1. You bet $1
2. You roll (always with a fair die)
3. Casino player rolls (maybe with fair die, maybe with loaded die)
4. Highest number wins $2

Question # 1 – Decoding

GIVEN

A sequence of rolls by the casino player

1245526462146146136136661664661636616366163616515615115146123562344

QUESTION

What portion of the sequence was generated with the fair die, and what portion with the loaded die?

This is the **DECODING** question in HMMs

Question # 2 – Evaluation

GIVEN

A sequence of rolls by the casino player

1245526462146146136136661664661636616366163616515615115146123562344

Prob = 1.3×10^{-35}

QUESTION

How likely is this sequence, given our model of how the casino works?

This is the **EVALUATION** problem in HMMs

Question # 3 – Learning

GIVEN

A sequence of rolls by the casino player

1245526462146146136136661664661636616366163616515615115146123562344

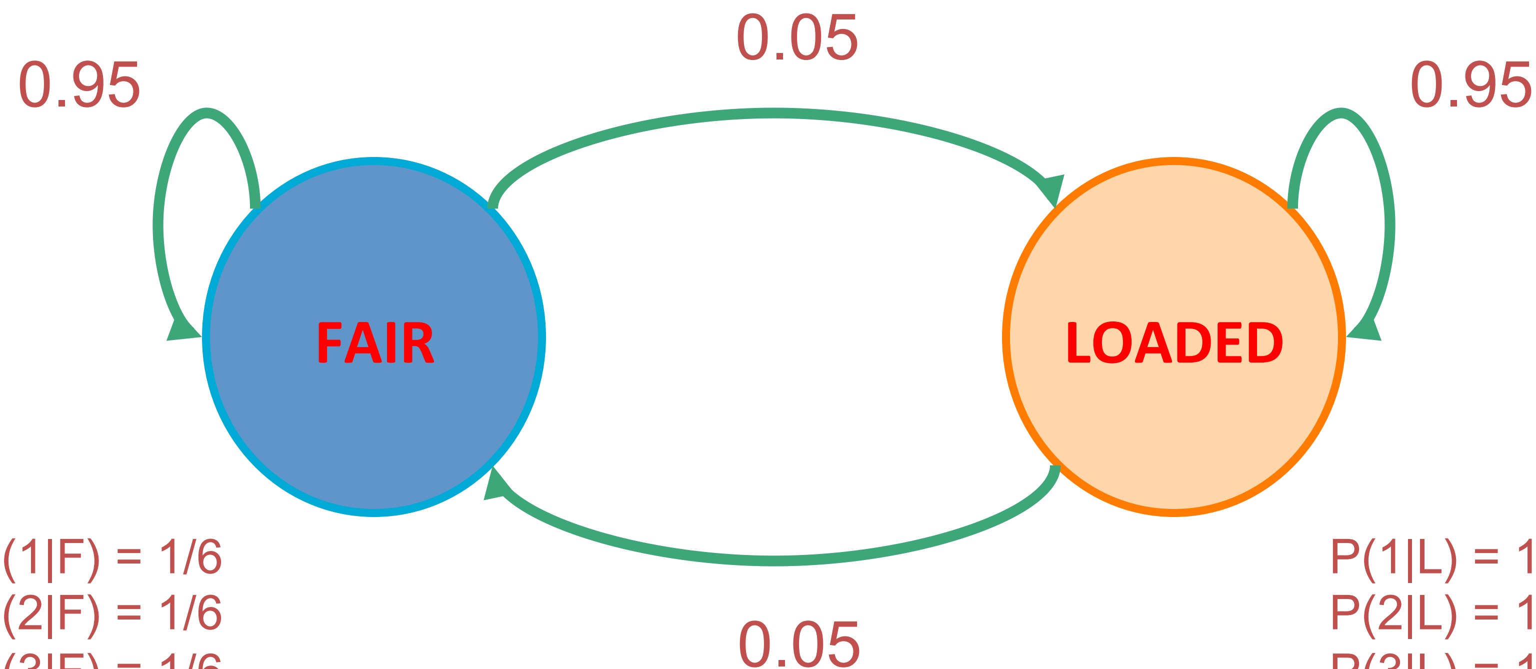
Prob(6) = 64%

QUESTION

How "loaded" is the loaded die? How "fair" is the fair die? How often does the casino player change from fair to loaded, and back?

This is the **LEARNING** question in HMMs

The dishonest casino model

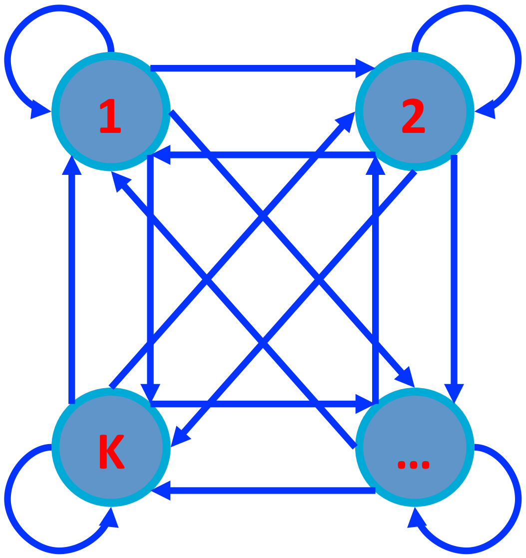


P(1|F) = 1/6
P(2|F) = 1/6
P(3|F) = 1/6
P(4|F) = 1/6
P(5|F) = 1/6
P(6|F) = 1/6

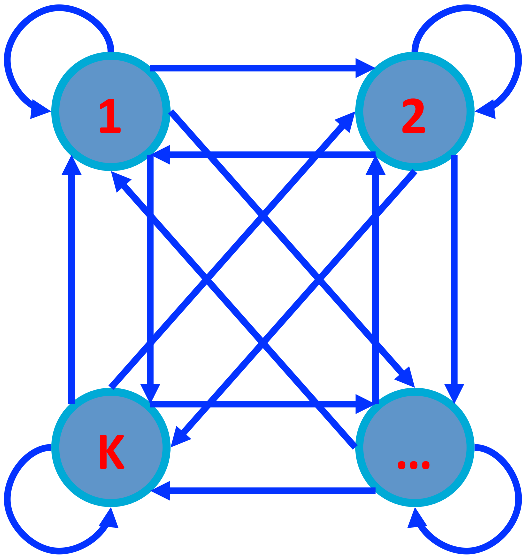
P(1|L) = 1/10
P(2|L) = 1/10
P(3|L) = 1/10
P(4|L) = 1/10
P(5|L) = 1/10
P(6|L) = 1/2

An HMM is memoryless

At each time step t,
the only thing that affects future states
is the current state π_t



An HMM is memoryless



At each time step t,

the only thing that affects future states

is the current state π_t

$P(\pi_{t+1} = k \mid \text{"whatever happened so far"}) =$

$P(\pi_{t+1} = k \mid \pi_1, \pi_2, \ldots, \pi_t, x_1, x_2, \ldots, x_t) =$

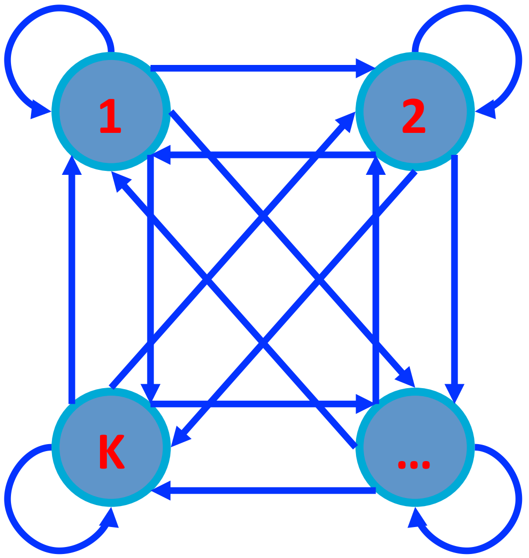
$P(\pi_{t+1} = k \mid \pi_t)$

An HMM is memoryless

At each time step t,

the only thing that affects x_t

is the current state π_t



$P(x_t = b \mid \text{"whatever happened so far"}) =$

$P(x_t = b \mid \pi_1, \pi_2, \ldots, \pi_t, x_1, x_2, \ldots, x_{t-1}) =$

$P(x_t = b \mid \pi_t)$

Definition of a hidden Markov model

Definition: A hidden Markov model (HMM)

- Alphabet $\Sigma = \{ b_1, b_2, \ldots, b_M \}$
- Set of states $Q = \{ 1, ..., K \}$
- Transition probabilities between any two states

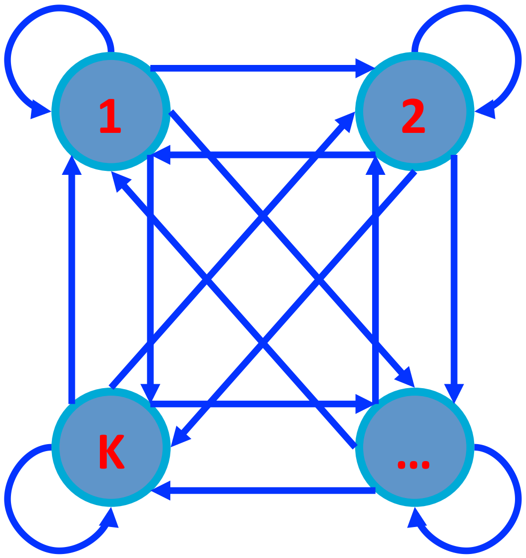
  a_{ij} = transition prob from state i to state j
  $a_{i1} + \ldots + a_{iK} = 1$, for all states i = 1…K

- Start probabilities a_{0i}

  $a_{01} + \ldots + a_{0K} = 1$

- Emission probabilities within each state

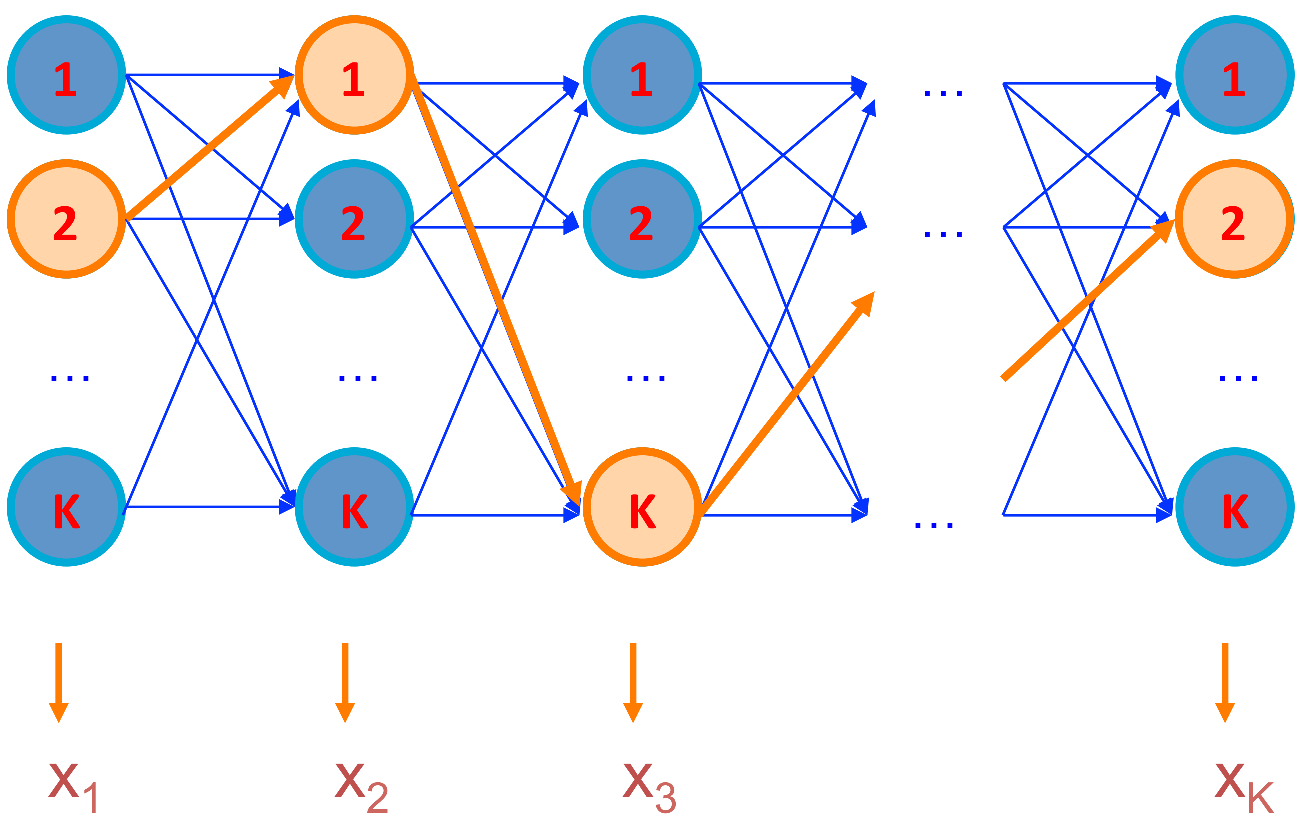
  $e_i(b) = P(x_i = b \mid \pi_i = k)$
  $e_i(b_1) + \ldots + e_i(b_M) = 1$, for all states i = 1…K



A parse of a sequence

Given a sequence $x = x_1 \ldots\ldots x_N$,

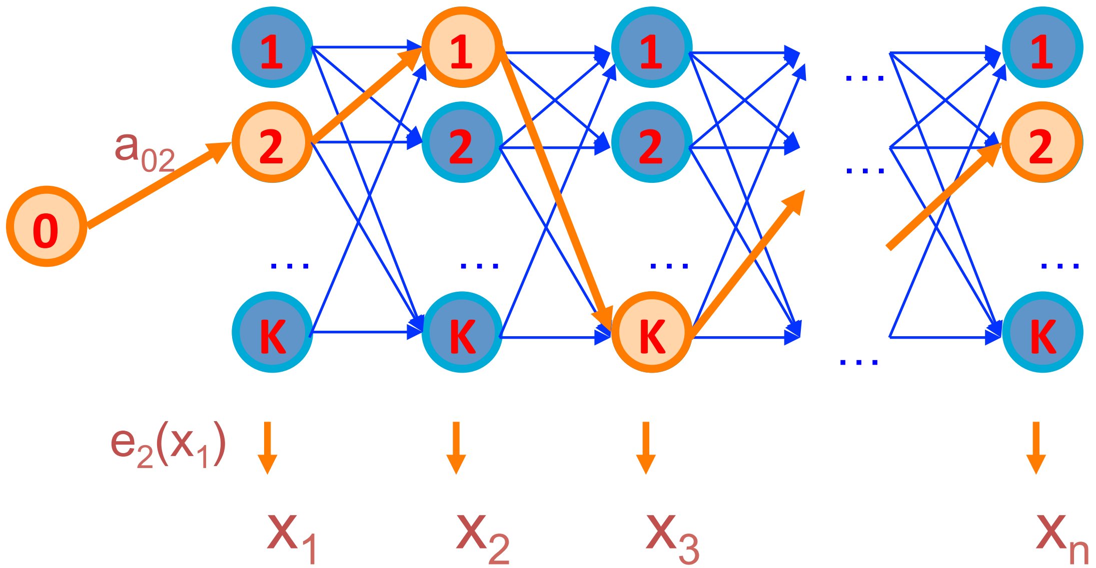
A <u>parse</u> of x is a sequence of states $\pi = \pi_1, \ldots\ldots, \pi_N$



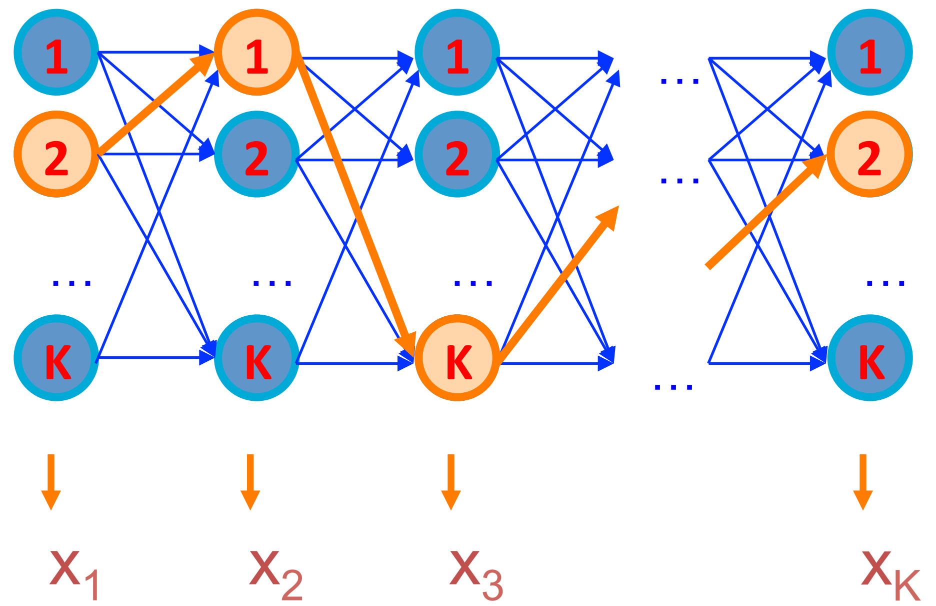
Generating a sequence by the model

Given a HMM, we can generate a sequence of length n as follows:

1. Start at state π_1 according to prob $a_{0\pi 1}$
2. Emit letter x_1 according to prob $e_{\pi 1}(x_1)$
3. Go to state π_2 according to prob $a_{\pi 1 \pi 2}$
4. … until emitting x_n



Likelihood of a parse



Given a sequence $x = x_1……x_N$
and a parse $\pi = \pi_1, ……, \pi_N$,

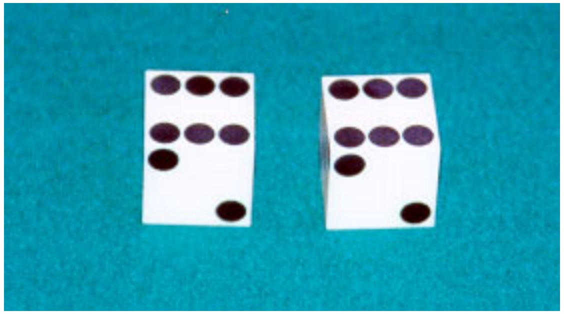
To find how likely this scenario is:
(given our HMM)

$$P(x, \pi) = P(x_1, …, x_N, \pi_1, ……, \pi_N) =$$

$$P(x_N \mid \pi_N)\, P(\pi_N \mid \pi_{N-1})\, ……P(x_2 \mid \pi_2)\, P(\pi_2 \mid \pi_1)\, P(x_1 \mid \pi_1)\, P(\pi_1) =$$

$$a_{0\pi 1}\, a_{\pi 1\pi 2}……a_{\pi N\text{-}1\pi N}\, e_{\pi 1}(x_1)……e_{\pi N}(x_N)$$

Example: the dishonest casino



Let the sequence of rolls be:

x = 1, 2, 1, 5, 6, 2, 1, 5, 2, 4

Then, what is the likelihood of

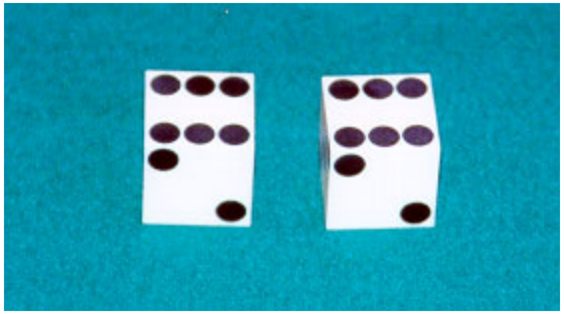
π = Fair, Fair, Fair, Fair, Fair, Fair, Fair, Fair, Fair, Fair?

(say initial probs $a_{0Fair} = ½$, $a_{oLoaded} = ½$)

$½ \times P(1 \mid Fair)\ P(Fair \mid Fair)\ P(2 \mid Fair)\ P(Fair \mid Fair) \ldots P(4 \mid Fair) =$

$½ \times (1/6)^{10} \times (0.95)^9 = .00000000521158647211 \sim= 0.5 \times 10^{-9}$

Example: the dishonest casino



So, the likelihood the die is fair in this run is just 0.521×10^{-9}

What is the likelihood of

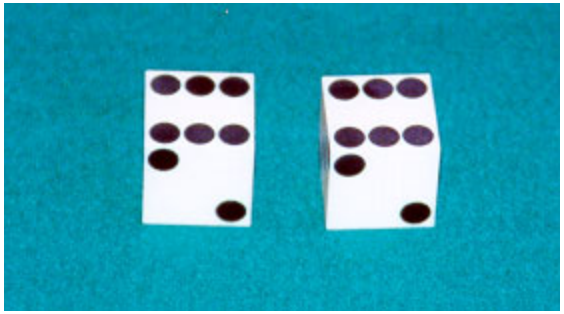
π = Loaded, Loaded, Loaded, Loaded, Loaded, Loaded, Loaded, Loaded, Loaded, Loaded?

$\frac{1}{2} \times P(1 \mid \text{Loaded})\, P(\text{Loaded}, \text{Loaded}) \ldots P(4 \mid \text{Loaded}) =$

$\frac{1}{2} \times (1/10)^9 \times (1/2)^1\, (0.95)^9 = .00000000015756235243 \sim= 0.16 \times 10^{-9}$

Therefore, it's somewhat more likely that all the rolls are done with the fair die, than that they are all done with the loaded die

Example: the dishonest casino



Let the sequence of rolls be:

x = 1, 6, 6, 5, 6, 2, 6, 6, 3, 6

Now, what is the likelihood π = F, F, …, F?

$\frac{1}{2} \times (1/6)^{10} \times (0.95)^9$ ~= 0.5×10^{-9}, same as before

What is the likelihood

π = L, L, …, L?

$\frac{1}{2} \times (1/10)^4 \times (1/2)^6 (0.95)^9 = .00000049238235134735$ ~= 0.5×10^{-7}

So, it is 100 times more likely the die is loaded

The three main questions on HMMs

1. **Decoding**

   GIVEN a HMM M, and a sequence x,
   FIND the sequence π of states that maximizes P[x, π | M]

2. **Evaluation**

   GIVEN a HMM M, and a sequence x,
   FIND Prob[x | M]

3. **Learning**

   GIVEN a HMM M, with unspecified transition/emission probs.,
   and a sequence x,
   FIND parameters $\theta = (e_i(.), a_{ij})$ that maximize P[x | θ]

Problem 1: Decoding

Find the most likely parse of a sequence

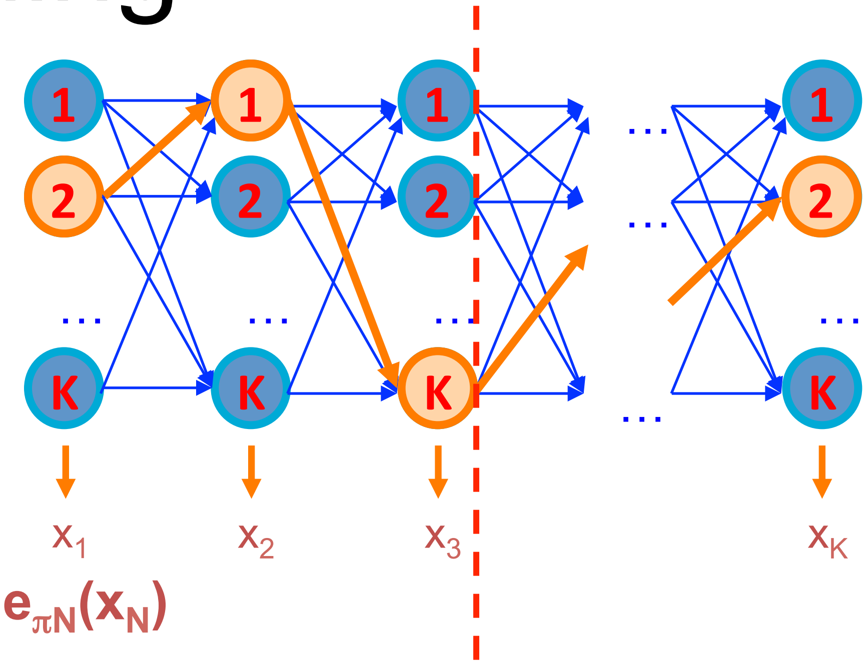
Decoding

GIVEN $x = x_1 x_2 \ldots\ldots x_N$

Find $\pi = \pi_1, \ldots\ldots, \pi_N,$
to maximize $P[\, x, \pi \,]$

$\pi^* = \text{argmax}_\pi \, P[\, x, \pi \,]$

Maximizes $a_{0\pi 1}\, e_{\pi 1}(x_1)\, a_{\pi 1 \pi 2} \ldots\ldots a_{\pi N\text{-}1 \pi N}\, e_{\pi N}(x_N)$



Dynamic Programming!

Given that we end up in state k at step i, maximize product to the left and right

$V_k(i) = \max_{\{\pi 1 \ldots \pi i\text{-}1\}} P[x_1 \ldots x_{i-1}, \pi_1, \ldots, \pi_{i-1}, x_i, \pi_i = k]$

$= $ Prob. of most likely sequence of states ending at state $\pi_i = k$

Decoding – main idea

Induction: Given that for all states k, and for a fixed position i,

$$\mathbf{V_k(i) = max_{\{\pi 1 \ldots \pi i-1\}} P[x_1 \ldots x_{i-1}, \pi_1, \ldots, \pi_{i-1}, x_i, \pi_i = k]}$$

What is $V_l(i+1)$?

From definition,

$$\begin{aligned}
V_l(i+1) &= \max_{\{\pi 1 \ldots \pi i\}} P[\, x_1 \ldots x_i, \pi_1, \ldots, \pi_i, x_{i+1}, \pi_{i+1} = l \,] \\
&= \max_{\{\pi 1 \ldots \pi i\}} P(x_{i+1}, \pi_{i+1} = l \mid x_1 \ldots x_i, \pi_1, \ldots, \pi_i)\, P[x_1 \ldots x_i, \pi_1, \ldots, \pi_i] \\
&= \max_{\{\pi 1 \ldots \pi i\}} P(x_{i+1}, \pi_{i+1} = l \mid \pi_i\,)\, P[x_1 \ldots x_{i-1}, \pi_1, \ldots, \pi_{i-1}, x_i, \pi_i] \\
&= \max_k \,[P(x_{i+1}, \pi_{i+1} = l \mid \pi_i = k)\, \mathbf{max_{\{\pi 1 \ldots \pi i-1\}} P[x_1 \ldots x_{i-1}, \pi_1, \ldots, \pi_{i-1}, x_i, \pi_i = k]}] \\
&= \max_k \,[\, P(x_{i+1} \mid \pi_{i+1} = l\,)\, P(\pi_{i+1} = l \mid \pi_i = k)\, \mathbf{V_k(i)}\,] \\
&= e_l(x_{i+1}) \max_k a_{kl}\, \mathbf{V_k(i)}
\end{aligned}$$

The Viterbi Algorithm

Input: $x = x_1 \ldots\ldots x_N$

Initialization:

$V_0(0) = 1$ (0 is the imaginary first position)

$V_k(0) = 0$, for all $k > 0$

Iteration:

$V_j(i) = e_j(x_i) \times \max_k a_{kj} V_k(i - 1)$

$Ptr_j(i) = \text{argmax}_k a_{kj} V_k(i - 1)$

Termination:

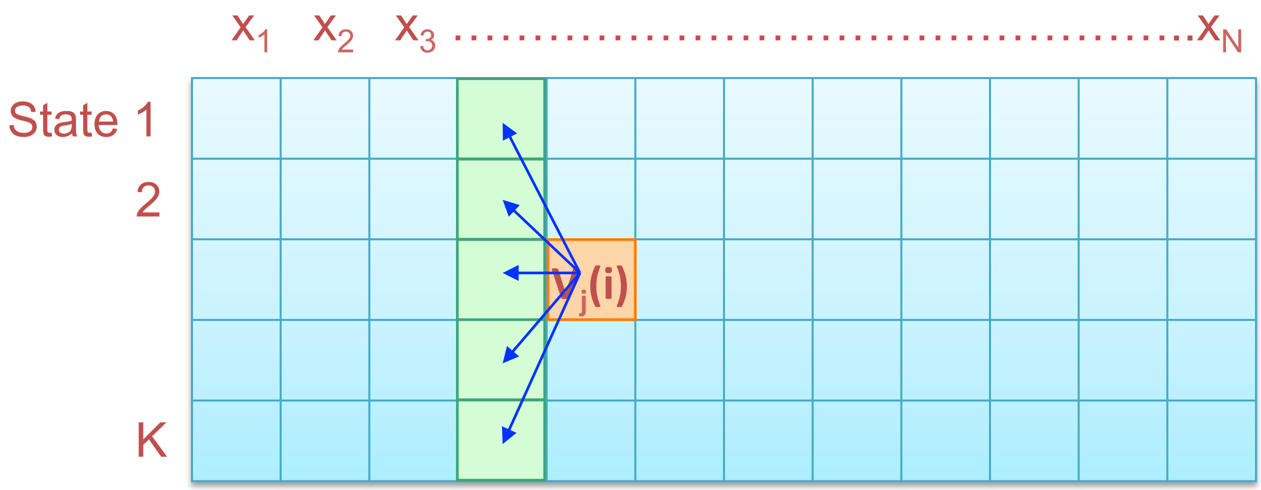
$P(x, \pi^*) = \max_k V_k(N)$

Traceback:

$\pi_N^* = \text{argmax}_k V_k(N)$

$\pi_{i-1}^* = Ptr_{\pi i}(i)$

The Viterbi Algorithm



Time:

O(K^2N)

Space:

O(KN)

Viterbi Algorithm – a practical detail

Underflows are a significant problem

$$P[\, x_1, \ldots, x_i, \pi_1, \ldots, \pi_i \,] = a_{0\pi 1}\, a_{\pi 1 \pi 2} \ldots\ldots a_{\pi i}\, e_{\pi 1}(x_1) \ldots\ldots e_{\pi i}(x_i)$$

These numbers become extremely small – underflow

Solution: Take the logs of all values

$$V_l(i) = \log e_k(x_i) + \max_k [\, V_k(i-1) + \log a_{kl} \,]$$

Example

Let x be a long sequence with a portion of ~ 1/6 6's,
followed by a portion of ~ ½ 6's…

x = 123456123456…12345 6626364656…1626364656

Then, it is not hard to show that optimal parse is (exercise):

FFF…………………...F LLL………………………...L

6 characters "123456" parsed as F, contribute $.95^6 \times (1/6)^6 = 1.6 \times 10^{-5}$
parsed as L, contribute $.95^6 \times (1/2)^1 \times (1/10)^5 = 0.4 \times 10^{-5}$

"162636" parsed as F, contribute $.95^6 \times (1/6)^6 = 1.6 \times 10^{-5}$
parsed as L, contribute $.95^6 \times (1/2)^3 \times (1/10)^3 = 9.0 \times 10^{-5}$

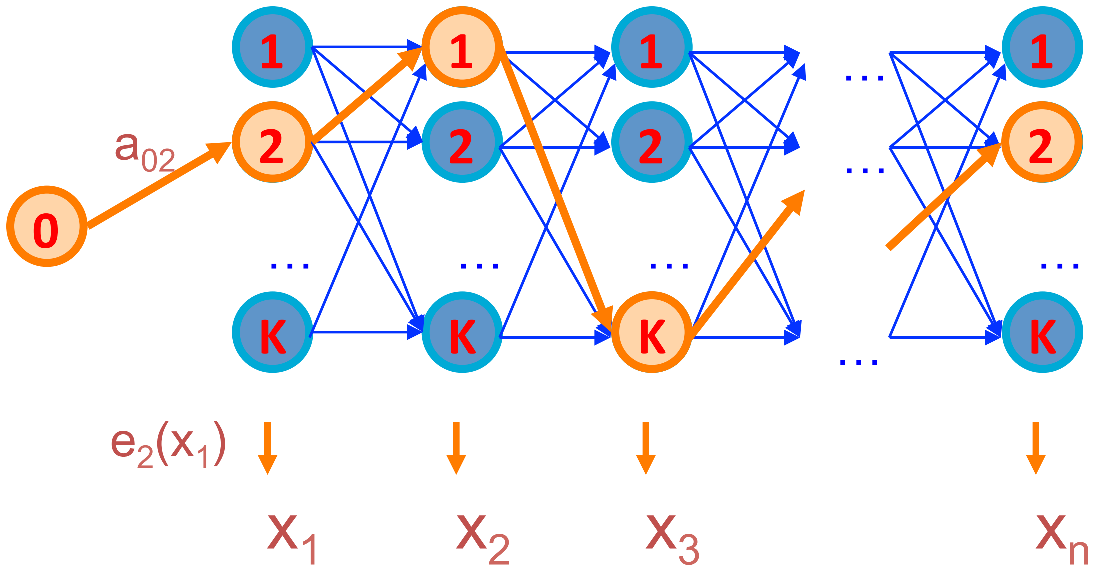
Problem 2: Evaluation

Compute the likelihood that a sequence is generated by the model

Generating a sequence by the model

Given a HMM, we can generate a sequence of length n as follows:

1. Start at state π_1 according to prob $a_{0\pi 1}$
2. Emit letter x_1 according to prob $e_{\pi 1}(x_1)$
3. Go to state π_2 according to prob $a_{\pi 1 \pi 2}$
4. … until emitting x_n



A couple of questions

Given a sequence x,

- What is the probability that x
- Given a position i, what is th

Example: the dishonest ca

P(box: FFFFFFFFFFF) =
$(1/6)^{11} * 0.95^{12}$ =
$2.76^{-9} * 0.54$ =
1.49^{-9}

P(box: LLLLLLLLLLL) =
$[(1/2)^6 * (1/10)^5] * 0.95^{10} * 0.05^2$ =
$1.56*10^{-7} \; * \; 1.5^{-3}$ =
0.23^{-9}

Say x = $\underbrace{12341\ldots231}_{F}\boxed{62616364616}\underbrace{234112\ldots21341}_{F}$

Most likely path: π = FF……F

(too "unlikely" to transition F → L → F)

However: marked letters more likely to be L than unmarked letters

Evaluation

We will develop algorithms that allow us to compute:

$P(x)$ Probability of x given the model

$P(x_i \ldots x_j)$ Probability of a substring of x given the model

$P(\pi_i = k \mid x)$ "Posterior" probability that the i^{th} state is k, given x

A more refined measure of which states x may be in

The Forward Algorithm

We want to calculate

$P(x)$ = probability of x, given the HMM

Sum over all possible ways of generating x:

$$P(x) = \Sigma_\pi P(x, \pi) = \Sigma_\pi P(x \mid \pi) P(\pi)$$

To avoid summing over an exponential number of paths π, define

$$f_k(i) = P(x_1 \ldots x_i, \pi_i = k)$$ (the forward probability)

"generate i first observations and end up in state k"

The Forward Algorithm – derivation

Define the forward probability:

$f_k(i) = P(x_1 \ldots x_i, \pi_i = k)$

$= \sum_{\pi 1 \ldots \pi i-1} P(x_1 \ldots x_{i-1}, \pi_1, \ldots, \pi_{i-1}, \pi_i = k)\, e_k(x_i)$

$= \sum_l \boldsymbol{\sum_{\pi 1 \ldots \pi i-2} P(x_1 \ldots x_{i-1}, \pi_1, \ldots, \pi_{i-2}, \pi_{i-1} = l)}\, a_{lk}\, e_k(x_i)$

$= \sum_l \boldsymbol{P(x_1 \ldots x_{i-1}, \pi_{i-1} = l)}\, a_{lk}\, e_k(x_i)$

$= e_k(x_i) \sum_l \boldsymbol{f_l(i - 1)}\, a_{lk}$

The Forward Algorithm

We can compute $f_k(i)$ for all k, i, using dynamic programming!

Initialization:

$f_0(0) = 1$

$f_k(0) = 0$, for all $k > 0$

Iteration:

$$f_k(i) = e_k(x_i) \sum_l f_l(i - 1)\, a_{lk}$$

Termination:

$$P(x) = \sum_k f_k(N)$$

Relation between Forward and Viterbi

VITERBI

Initialization:

$V_0(0) = 1$

$V_k(0) = 0$, for all $k > 0$

Iteration:

$$V_j(i) = e_j(x_i) \ \mathbf{max}_k \ V_k(i-1) \ a_{kj}$$

Termination:

$$P(x, \pi^*) = \mathbf{max}_k \ V_k(N)$$

FORWARD

Initialization:

$f_0(0) = 1$

$f_k(0) = 0$, for all $k > 0$

Iteration:

$$f_l(i) = e_l(x_i) \ \Sigma_k \ f_k(i-1) \ a_{kl}$$

Termination:

$$P(x) = \Sigma_k \ f_k(N)$$

Motivation for the Backward Algorithm

We want to compute

$P(\pi_i = k \mid x)$,

the probability distribution on the i^{th} position, given x

We start by computing

$$
\begin{aligned}
P(\pi_i = k, x) &= P(x_1 \ldots x_i, \pi_i = k, x_{i+1} \ldots x_N) \\
&= P(x_1 \ldots x_i, \pi_i = k)\, P(x_{i+1} \ldots x_N \mid x_1 \ldots x_i, \pi_i = k) \\
&= \boxed{P(x_1 \ldots x_i, \pi_i = k)}\, \boxed{P(x_{i+1} \ldots x_N \mid \pi_i = k)}
\end{aligned}
$$

Forward, $f_k(i)$ Backward, $b_k(i)$

Then, $P(\pi_i = k \mid x) = P(\pi_i = k, x) / P(x)$

The Backward Algorithm – derivation

Define the backward probability:

$b_k(i) = P(x_{i+1} \ldots x_N \mid \pi_i = k)$ *"starting from i^{th} state = k, generate rest of x"*

$= \sum_{\pi_{i+1} \ldots \pi_N} P(x_{i+1}, x_{i+2}, \ldots, x_N, \pi_{i+1}, \ldots, \pi_N \mid \pi_i = k)$

$= \sum_l \sum_{\pi_{i+1} \ldots \pi_N} P(x_{i+1}, x_{i+2}, \ldots, x_N, \pi_{i+1} = l, \pi_{i+2}, \ldots, \pi_N \mid \pi_i = k)$

$= \sum_l e_l(x_{i+1})\, a_{kl} \boldsymbol{\sum_{\pi_{i+1} \ldots \pi_N} P(x_{i+2}, \ldots, x_N, \pi_{i+2}, \ldots, \pi_N \mid \pi_{i+1} = l)}$

$= \sum_l e_l(x_{i+1})\, a_{kl}\, \mathbf{b_l(i+1)}$

The Backward Algorithm

We can compute $b_k(i)$ for all k, i, using dynamic programming

Initialization:

$b_k(N) = 1$, for all k

Iteration:

$b_k(i) = \Sigma_l \, e_l(x_{i+1}) \, a_{kl} \, b_l(i+1)$

Termination:

$P(x) = \Sigma_l \, a_{0l} \, e_l(x_1) \, b_l(1)$

Computational Complexity

What is the running time, and space required, for Forward and Backward?

Time: $O(K^2N)$

Space: $O(KN)$

Useful implementation technique to avoid underflows

Viterbi: sum of logs

Forward/Backward: rescaling at each few positions by multiplying by a constant

Posterior Decoding

We can now calculate

$$P(\pi_i = k \mid x) = \frac{f_k(i)\, b_k(i)}{P(x)}$$

Then, we can ask

What is the most likely state at position i of sequence x:

Define $\pi^\wedge$ by Posterior Decoding:

$$\pi^\wedge_i = \text{argmax}_k\, P(\pi_i = k \mid x)$$

$P(\pi_i = k \mid x) =$

$P(\pi_i = k\ ,\ x)/P(x) =$

$P(x_1, \ldots, x_i, \pi_i = k, x_{i+1}, \ldots\, x_n)\ /\ P(x) =$

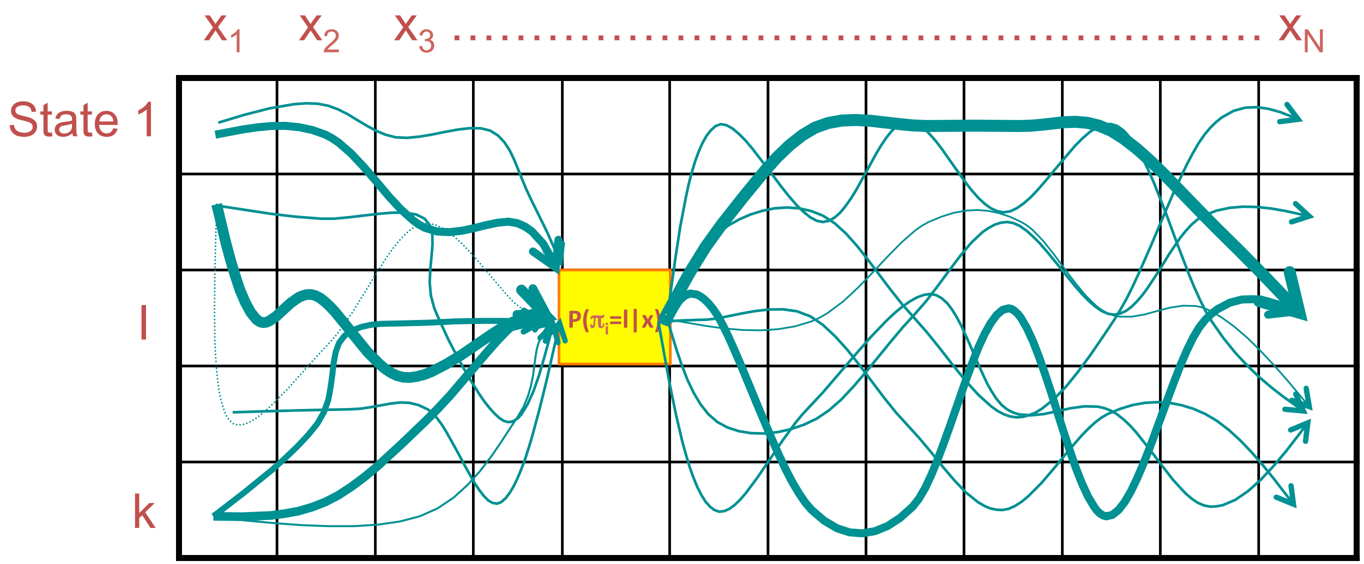
$P(x_1, \ldots, x_i, \pi_i = k)\ P(x_{i+1}, \ldots\, x_n \mid \pi_i = k)\ /\ P(x) =$

$f_k(i)\ b_k(i)\ /\ P(x)$

Posterior Decoding

- For each state,
  - Posterior Decoding gives us a curve of likelihood of state for each position
  - That is sometimes more informative than Viterbi path π^*
- Posterior Decoding may give an invalid sequence of states (of probability 0)
  - Why?

Posterior Decoding



- $P(\pi_i = k \mid x) = \sum_{\pi} P(\pi \mid x)\, \mathbf{1}(\pi_i = k)$

  $= \sum_{\{\pi:\pi[i] = k\}} P(\pi \mid x)$

$\mathbf{1}(\psi) = 1$, if ψ is true
0, otherwise

Viterbi, Forward, Backward

VITERBI

Initialization:

$V_0(0) = 1$

$V_k(0) = 0$, for all $k > 0$

Iteration:

$V_l(i) = e_l(x_i)\ \mathbf{max_k}\ V_k(i-1)\ a_{kl}$

Termination:

$P(x, \pi^*) = \mathbf{max_k}\ V_k(N)$

FORWARD

Initialization:

$f_0(0) = 1$

$f_k(0) = 0$, for all $k > 0$

Iteration:

$f_l(i) = e_l(x_i)\ \boldsymbol{\Sigma_k}\ f_k(i-1)\ a_{kl}$

Termination:

$P(x) = \boldsymbol{\Sigma_k}\ f_k(N)$

BACKWARD

Initialization:

$b_k(N) = 1$, for all k

Iteration:

$b_l(i) = \boldsymbol{\Sigma_k}\ e_l(x_i+1)\ a_{kl}\ b_k(i+1)$

Termination:

$P(x) = \boldsymbol{\Sigma_k}\ a_{0k}\ e_k(x_1)\ b_k(1)$

Problem 3: Learning

Find the parameters that maximize the likelihood of the observed sequence

Estimating HMM parameters

- Easy if we know the sequence of hidden states
  - Count # times each transition occurs
  - Count #times each observation occurs in each state
- Given an HMM and observed sequence, we can compute the distribution over paths, and therefore the expected counts
- “Chicken and egg” problem

Solution: Use the EM algorithm

- Guess initial HMM parameters
- **E step:** Compute distribution over paths
- **M step:** Compute max likelihood parameters
- But how do we do this efficiently?

The forward-backward algorithm

- Also known as the Baum-Welch algorithm
- Compute probability of each state at each position using forward and backward probabilities
  → (Expected) observation counts
- Compute probability of each pair of states at each pair of consecutive positions *i* and *i+1* using *forward(i)* and *backward(i+1)*
  $\text{Count}(k \rightarrow l) = \Sigma_i f_k(i)\, a_{kl}\, b_l(i+1) / P(x)$
  → (Expected) transition counts